Recent Results in Lattice QCD

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QCD – Gauge theory of the strong interaction

- Lagrangian: formulated in terms of quarks and gluons

$$\mathcal{L}_{\rm QCD} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_f \overline{\psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \psi_f, \quad f = u, d, s, c, b, t$$

$$D_\mu = \partial_\mu - ig(\tfrac{1}{2}\lambda^a) A^a_\mu$$

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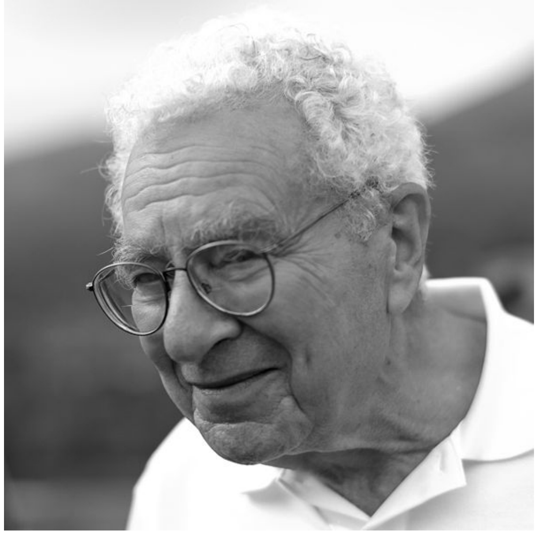
Harald Fritzsch



Murray Gell-Mann

Heinrich Leutwyler

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$$D_\mu = \partial_\mu - ig(\tfrac{1}{2}\lambda^a) A^a_\mu$$

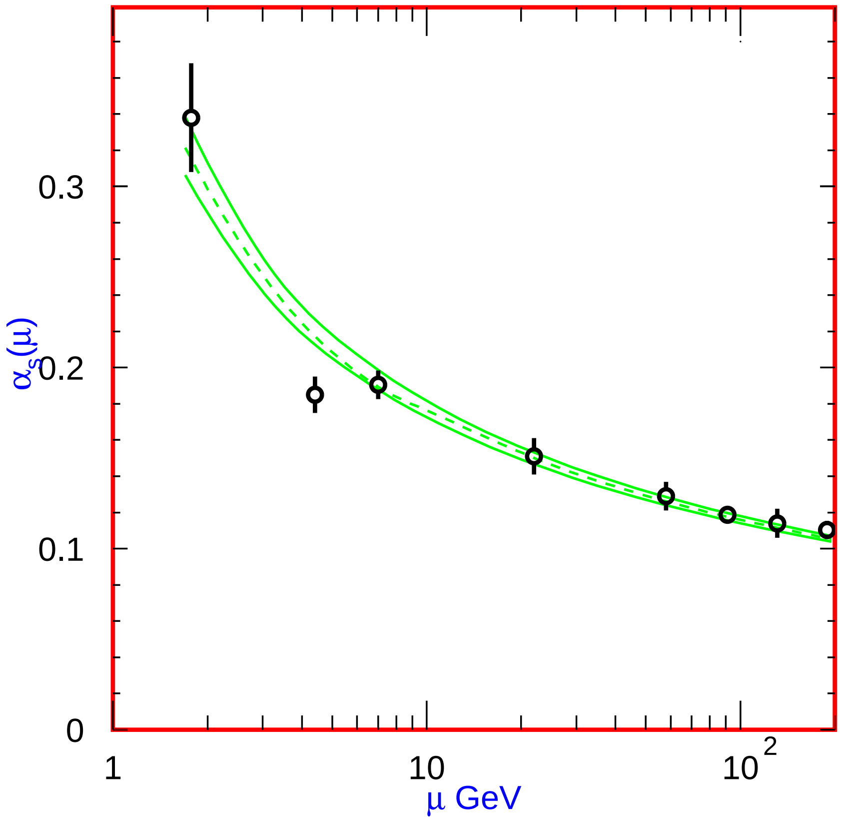
- Parameters:

gauge coupling: $g, \quad \alpha_s = g^2/4\pi$
quark masses: $m_u,\, m_d,\, m_s, \ldots$

- Amazingly simple structure

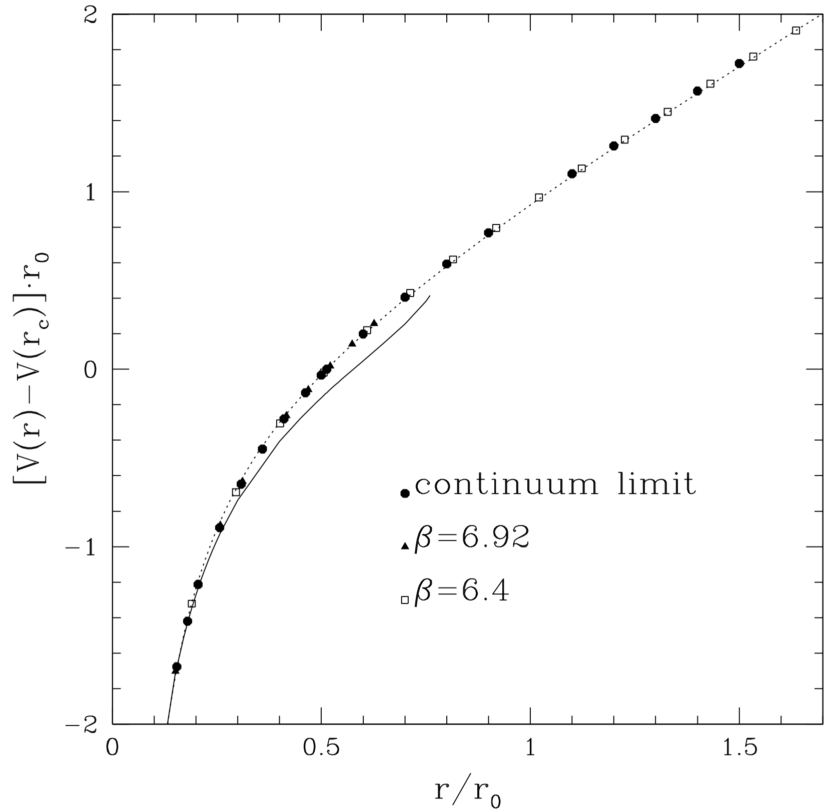
Properties of QCD

Asymptotic freedom: $g(\mu)$



[Yao et al., PDG 2006]

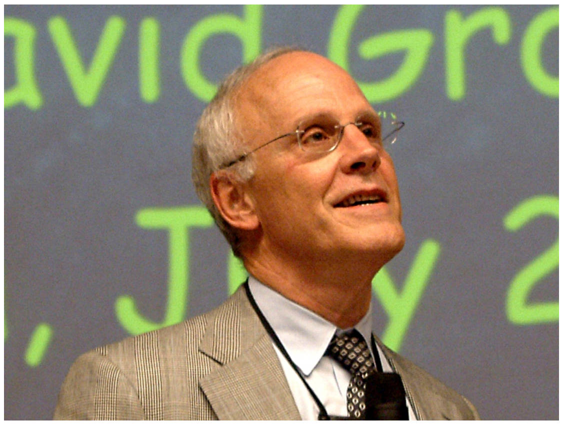
Confinement



[Necco & Sommer, Nucl Phys B622 (2002) 328]

Nobel Prize in Physics 2004

"...for the discovery of asymptotic freedom in the theory of the strong interaction"



David Gross

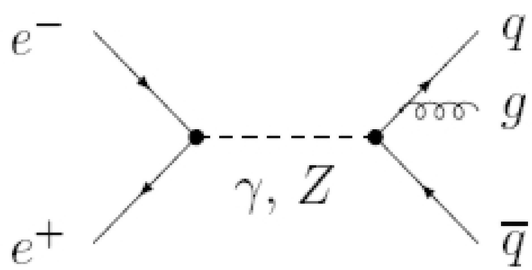
Frank Wilczek

David Politzer

(c.f. D. Politzer: *The Dilemma of Attribution* — Nobel lecture)

$\mu \simeq 100\,\mathrm{GeV}$: weakly coupled quarks and gluons:

$e^+e^- \to$ jets

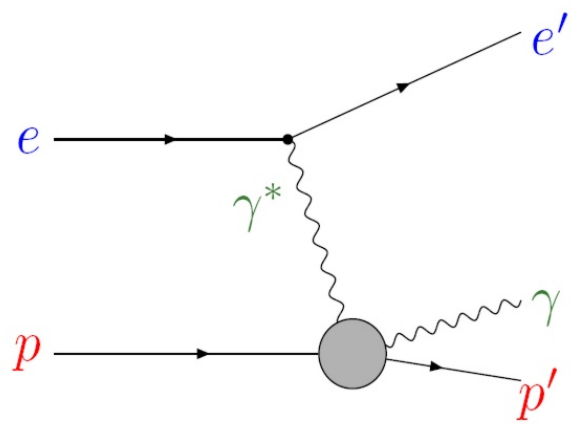


$\alpha_s(M_Z) = 0.121 \pm 0.007$

$\mu \simeq 1\,\mathrm{GeV}$: bound states of quarks and gluons:

$$\pi, K, \ldots, \rho, K^*, \ldots, P, \Sigma, \ldots, \Delta, \Sigma^*, \ldots, G(?)$$

$e\,p \to e\,p\,\gamma$



Perturbation theory in α_s not applicable!

→ Connecting low- and high-energy regimes of the strong interaction requires non-perturbative treatment

Lattice QCD *[Wilson 1974]*

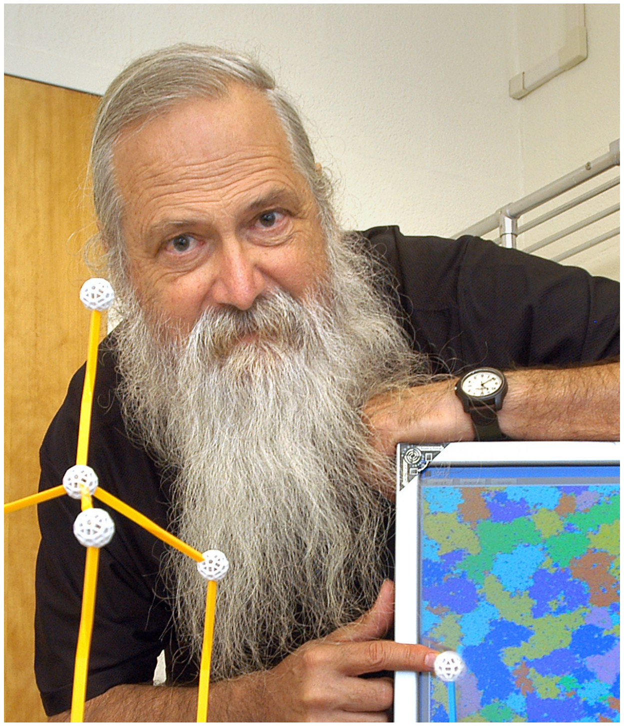
→ Formulation of QCD on discretised space-time

→ Connecting low- and high-energy regimes of the strong interaction requires non-perturbative treatment

Lattice QCD *[Wilson 1974]*

→ Formulation of QCD on discretised space-time

→ Determination of observables using numerical simulations *[Creutz 1979ff]*



Outline:

Lattice QCD — The Method

1. Lattice actions for QCD
2. Path integral & observables
3. Algorithms & machines

Lattice — Recent Results

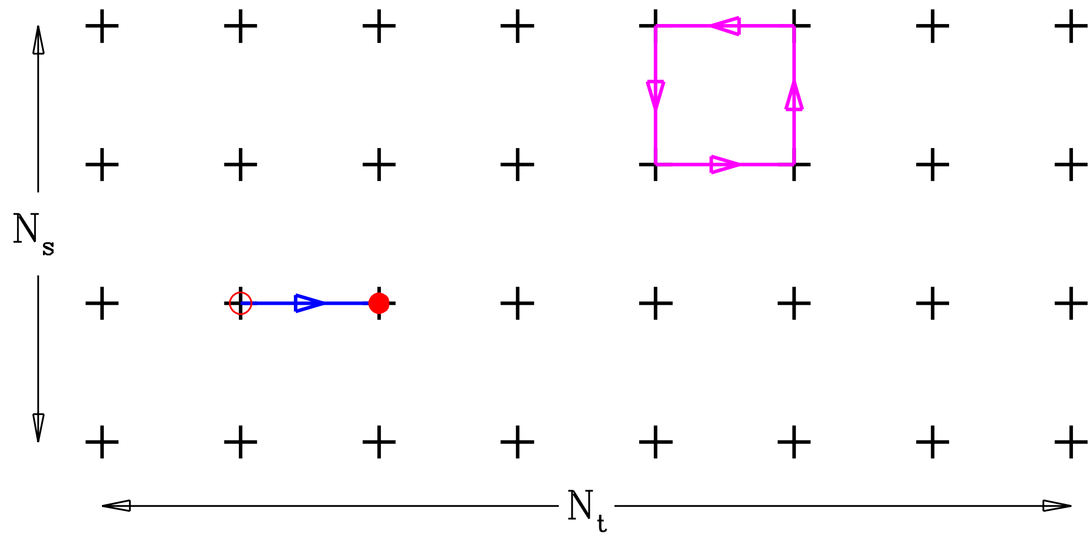
4. Hadron spectroscopy
5. Kaon physics
6. Hadronic vacuum polarisation and $(g-2)_\mu$

Lattice QCD — The Method

1. Basic Concepts of Lattice QCD

Minkowski space-time, continuum $\longrightarrow$ Euclidean space-time, discretised

Lattice spacing $a, \quad a^{-1} \sim \Lambda_{\mathrm{UV}}, \quad x_\mu = n_\mu a$
Finite volume $L^3 \cdot T, \quad N_s = L/a, \quad N_t = T/a$



(anti)quarks:	$\psi(x)$, $\overline{\psi}(x)$	lattice sites
gluons:	$U_\mu(x) = \mathrm{e}^{aA_\mu(x)} \in \mathrm{SU}(3)$	links
field tensor:	$P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x)$	"plaquettes"

- Formulate expressions for the QCD action in terms of link variables and fermionic fields

Lattice action: $S[U,\overline{\psi},\psi] = S_{\rm G}[U] + S_{\rm F}[U,\overline{\psi},\psi]$

Wilson "plaquette" action for Yang-Mills theory *[Wilson 1974]*

$$S_{\rm G}[U] = \beta \sum_x \sum_{\mu<\nu} \left(1 - \tfrac{1}{3}{\rm Re}\,\mathsf{Tr}\,P_{\mu\nu}(x)\right), \quad \beta = 6/g_0^2, \quad \text{(gauge invariant)}$$

$$P_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)$$

- For small lattice spacings:

$$S_{\rm G}[U] \longrightarrow -\frac{1}{2g_0^2}\int {\rm d}^4x\,\mathsf{Tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] + {\rm O}(a)$$

- **N.B.** Discretisation not unique!

Fermionic part:

- Discretised version of the covariant derivative:

$$\nabla_\mu \psi(x) \equiv \frac{1}{a}\Big(U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)\Big)$$
$$\nabla^*_\mu \psi(x) \equiv \frac{1}{a}\Big(\psi(x) - U^\dagger_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})\Big)$$

- "Naive" discretisation of fermionic part $S_{\rm F}$:

$$D_{\rm naive} + m_f = \frac{1}{2}\gamma_\mu \left(\nabla_\mu + \nabla^*_\mu\right) + m_f$$

$$\widetilde{D}_{\rm naive}(p) = i\gamma_\mu \frac{1}{a}\sin(ap_\mu) = i\gamma_\mu p_\mu + {\rm O}(a^2) \quad \text{(free theory)}$$

$\longrightarrow$ $\widetilde{D}_{\rm naive}(p)$ vanishes for $p_\mu = 0, \pi/a$

$\longrightarrow$ produces $2^4 = 16$ poles in fermion propagator of flavour f

$\longrightarrow$ 16-fold degeneracy of fermion spectrum:

Fermion doubling problem

Fermionic discretisations:

a. Wilson fermions *[Wilson 1974/75]*

b. Staggered (Kogut-Susskind) fermions *[Kogut+Susskind 1975]*

c. Overlap/Domain Wall fermions *[Kaplan '92, Furman+Shamir '96, Neuberger '98]*

d. "Perfect"/Fixed point actions *[Hasenfratz+Niedermaier '93/'98]*

c.+d.: Ginsparg-Wilson fermions *[Ginsparg+Wilson 1982, Lüscher 1998]*

e. Minimally doubled fermions *[Karsten 1981, Wilczek 1987, Creutz 2007, Boriçi 2008]*

Wilson fermions

- Add a term to $D_{\rm naive}$ which formally vanishes as $a \to 0$:

$$D_{\rm w} + m_f = \frac{1}{2}\gamma_\mu \left(\nabla_\mu + \nabla_\mu^*\right) + ar\nabla_\mu^*\nabla_\mu + m_f$$

$$\widetilde{D}_{\rm w}(p) = i\gamma_\mu \frac{1}{a}\sin(ap_\mu) + \frac{2r}{a}\sin^2\left(\frac{ap_\mu}{2}\right) \quad \text{(free theory)}$$

⇒ Mass of doubler states receives contribution $\propto r/a$: pushed to cutoff scale

:–) Complete lifting of degeneracy

:–(Explicit breaking of chiral symmetry:

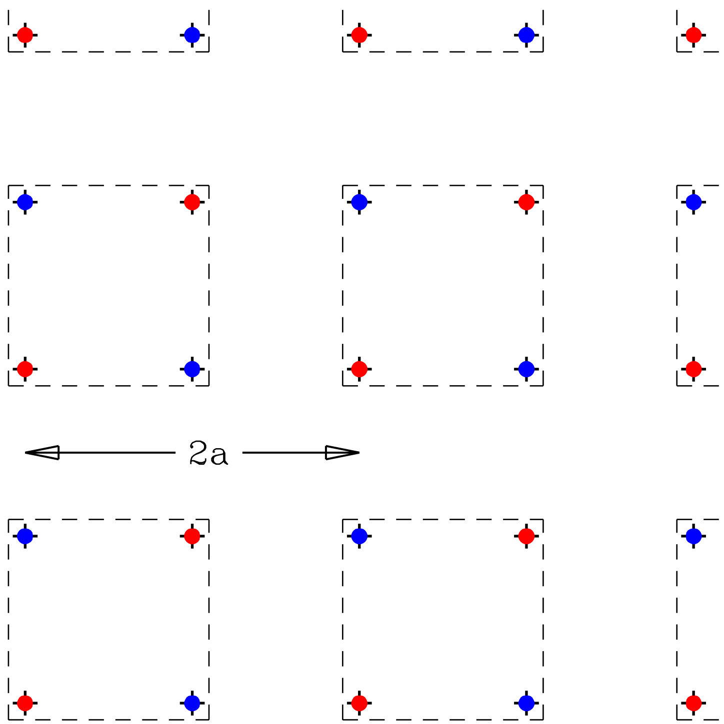
even for $m_f = 0$ the action is no longer invariant under

$$\psi(x) \to {\rm e}^{i\alpha\gamma_5}\psi(x), \qquad \overline{\psi}(x) \to \overline{\psi}(x){\rm e}^{i\alpha\gamma_5}$$

Mostly acceptable, but makes things more complicated

Staggered (Kogut-Susskind) fermions

- Reduce d.o.f. by distributing single spinor components over corners of hypercube



:–(Only partial lifting of degeneracy:

$$16 \longrightarrow 4$$

$\Rightarrow$ 4 "tastes" per physical flavour

- Flavour symmetry broken: gluons mix "tastes"

:–) Remnant of chiral symmetry: global $\mathrm{U}(1) \otimes \mathrm{U}(1)$

Chiral Symmetry on the Lattice

- Lattice regularisation: incompatible with chiral symmetry?

  Either: fermion doubling problem

  Or : explicit chiral symmetry breaking: $\{\gamma_5, D\} \neq 0$

  [Nielsen+Ninomiya 1979]

- Chiral symmetry at non-zero lattice spacing realised if

  [Ginsparg & Wilson 1982, Lüscher 1998]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

- Explicit construction: Neuberger-Dirac operator *[Neuberger 1998]*

$$D_{\rm N} = \frac{1}{a}\left\{1 - \frac{A}{\sqrt{A^\dagger A}}\right\}, \quad A = 1 - aD_{\rm w}$$

$D_{\rm w}$: massless Wilson-Dirac operator

$$S_{\mathrm{F}}[U,\overline{\psi},\psi] = a^4 \sum_x \overline{\psi}(x)[D_{\mathrm{N}}\psi](x)$$ – No fermion doublers!

- Invariance under infinitesimal chiral transformations:

$$\psi \to \psi + \epsilon\delta\psi, \quad \delta\psi = \gamma_5(1-\tfrac{1}{2}aD)\psi$$
$$\overline{\psi} \to \overline{\psi} + \delta\overline{\psi}\epsilon, \quad \delta\overline{\psi} = \overline{\psi}(1-\tfrac{1}{2}aD)\gamma_5$$

- D_{N} satisfies the Atiyah-Singer index theorem:
  [Hasenfratz, Laliena & Niedermayer 1998; Neuberger 1998]

$$\mathrm{index}(D_{\mathrm{N}}) = a^5 \sum_x \frac{1}{2}\,\mathrm{Tr}\,(\gamma_5 D_{\mathrm{N}}) = n_- - n_+$$

$\longrightarrow$ D_{N} exhibits $|n_- - n_+|$ exact zero modes

- But: numerical implementation of D_{N} expensive

Minimally doubled fermions

- Boriçi-Creutz construction: *[Creutz 2007-08; Boriçi 2008]*

$$\widetilde{D}(p) = \frac{i}{a}\sum_\mu \gamma_\mu \sin(ap_\mu) + \frac{i}{a}\sum_\mu \gamma'_\mu \cos(ap_\mu) + 2i\Gamma$$

$$\Gamma = \frac{1}{2}\sum_\mu \gamma_\mu, \qquad \gamma'_\mu = \Gamma\gamma_\mu\Gamma$$

→ two poles at $ap_\mu = (0,0,0,0)$ and $(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$,
corresponding to a pair of quarks of opposite chirality: ideal for $N_\mathrm{f} = 2$

- Strictly local; one exact chiral symmetry — but: hypercubic symmetry broken

- Lattice perturbation theory at one loop: *[Capitani, Creutz, Weber, H.W., arXiv:1006.2009]*
  - simple expressions for conserved vector- and axial-vector currents
  - hypercubic symmetry breaking:
    mixing with two dimension-four and one dimension-three operators
  - must tune the coefficients of three counterterms to the bare actions

2. Path integral and observables

Lattice formulation . . .

. . . preserves gauge invariance

. . . defines observables without reference to perturbation theory

. . . allows for stochastic evaluation of observables

Expectation value:

$$\begin{aligned}\langle\Omega\rangle &= \frac{1}{Z}\int D[U]D[\overline{\psi}]D[\psi]\,\Omega\,\mathrm{e}^{-S_{\mathrm{G}}[U]-S_{\mathrm{F}}[U,\overline{\psi},\psi]} \\ &= \frac{1}{Z}\int D[U]\,\Omega\,\prod_f \det\left(\gamma_\mu D_\mu + m_f\right)\,\mathrm{e}^{-S_{\mathrm{G}}[U]} \\ &= \frac{1}{Z}\int \prod_{x,\mu} dU_\mu(x)\,\Omega\,\prod_f \det\left(D_{\mathrm{lat}} + m_f\right)\,\mathrm{e}^{-S_{\mathrm{G}}[U]}\end{aligned}$$

Monte Carlo simulation

1. Generate set of N_c configurations of gauge fields $\{U_\mu(x)\}$, $i = 1, \ldots, N_c$, with probability distribution

$$W = \prod_f \det\left(D_{\rm lat} + m_f\right)\, \mathrm{e}^{-S_{\rm G}[U]}$$

   $\longrightarrow$ "Importance sampling"

$\longrightarrow$ Define an algorithm based on a Markov process:

   Generate sequence $\{U\}_1 \longrightarrow \{U\}_2 \longrightarrow \ldots \longrightarrow \{U\}_{N_c}$

   Transition probability given by W

2. Evaluate observable for configuration i

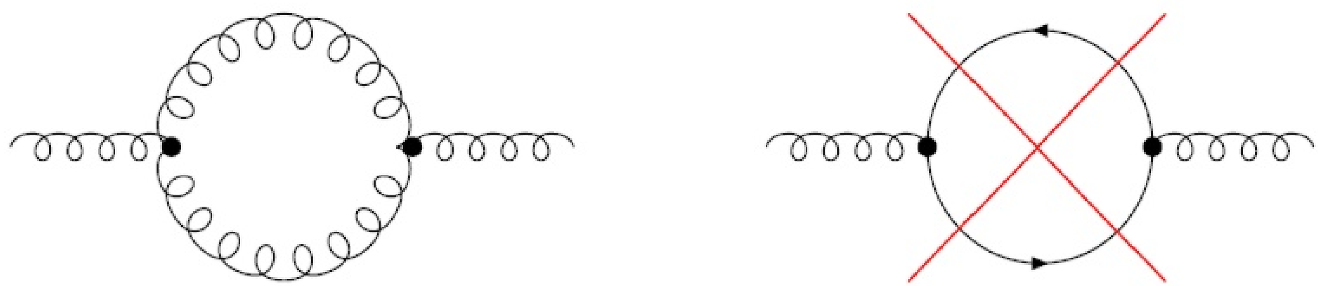
$$\overline{\Omega} = \frac{1}{N_c}\sum_{i=1}^{N_c} \Omega_i, \quad \langle\Omega\rangle = \lim_{N_c\to\infty} \overline{\Omega}, \qquad \text{statistical error: } \propto 1/\sqrt{N_c}$$

Dynamical quark effects

- $\det(D_{\text{lat}} + m_f)$: incorporates contributions of quark loops to $\langle\Omega\rangle$; non-local object; expensive to compute

- "Quenched Approximation:" $\det(D_{\text{lat}} + m_f) = 1$

$\longrightarrow$ Quark loops are entirely suppressed



- Inclusion of dynamical quark effects numerically expensive

Correlation functions

- Particle spectrum defined implicitly by correlation functions

- Consider K-Meson: $\phi_K(x) = s(x)\gamma_0\gamma_5\bar{u}(x)$

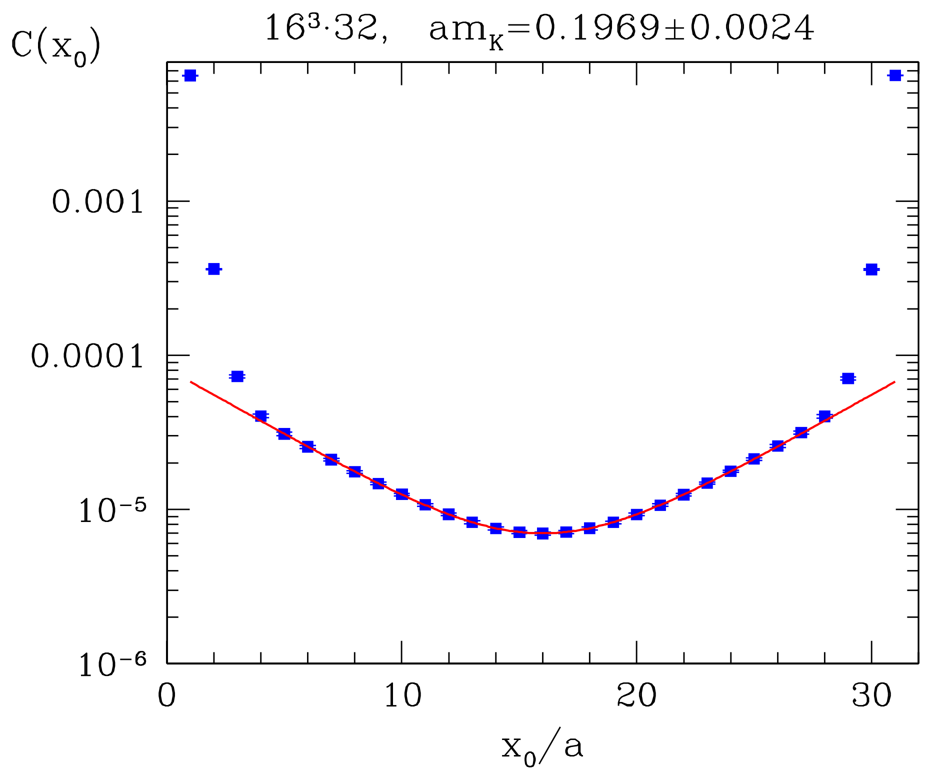
$$\sum_{\vec{x}} \left\langle \phi_K(x)\phi_K^\dagger(0) \right\rangle \overset{x_0 \gg 0}{\sim} \frac{|\langle 0|\phi_K|\mathrm{K}\rangle|^2}{2m_\mathrm{K}} \left\{ \mathrm{e}^{-m_\mathrm{K} x_0} + \mathrm{e}^{-m_\mathrm{K}[T-x_0]} \right\}$$

- Study asymptotic behaviour (large x_0) of Euclidean correlation functions:

$\longrightarrow$ exponential fall-off determines m_K

$\longrightarrow$ overall factor yields hadronic matrix element:

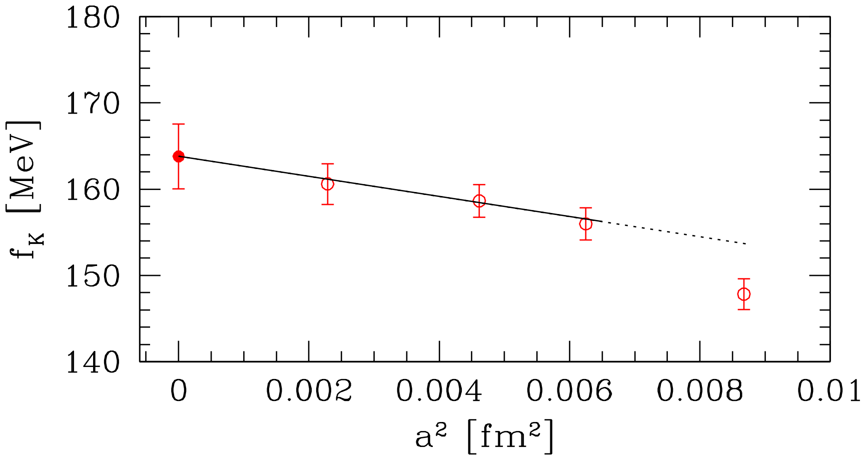
$$\langle 0|\phi_K|\mathrm{K}\rangle \propto f_\mathrm{K}$$



Continuum limit

$$\langle\Omega\rangle = \langle\Omega\rangle^{\mathsf{lat}} + O(a^p), \quad p \in \mathbb{N}, \qquad \text{lattice artefacts}$$

fermion discretisation	
Wilson	$\mathrm{O}(a)$
Improved Wilson	$\mathrm{O}(\alpha_s a)$, $\mathrm{O}(a^2)$
Staggered	$\mathrm{O}(a^2)$
DWF, Neuberger	$\mathrm{O}(a^2)$



[Garden, Heitger, Sommer, H.W. 1999]

- Classical continuum limit: $a \to 0$
- QFT: adjust the bare parameters as cutoff is removed, whilst keeping "constant physics"
- $$a\Lambda = \left(b_0 g_0^2\right)^{-b_1/2b_0^2} \mathrm{e}^{-1/2b_0 g_0^2} \ldots$$

  Continuum limit: $\beta = 6/g_0^2$

  $$a \to 0 \quad \Leftrightarrow \quad \beta \to \infty$$
- Perform simulations at several values of β and extrapolate to $a = 0$.

Eliminating the bare parameters

- Bare parameters: $g_0,\ m_u,\ m_d,\ m_s,\ldots$

- Can freely choose bare quark mass m_q in simulations;
  Which value of m_q corresponds to $m_u,\ m_d,\ldots$?

- Obtain hadron masses as functions of m_q, e.g. $am_{\rm PS}(m_{q_1},m_{q_2})$

- Quark mass dependence of hadron masses:

$$m_{\rm PS}^2 \propto m_q, \quad m_{\rm V},\, m_N \propto m_q$$

- Eliminate the bare parameters in favour of hadronic input quantities:

$$g_0 \sim 1/\ln a: \quad a^{-1}\,[\mathrm{GeV}] = \frac{Q\,[\mathrm{GeV}]}{(aQ)}, \quad Q = f_\pi, m_N, \Delta^{\Upsilon}_{\rm 1P-1S}, \ldots$$

$$\hat{m} = \tfrac{1}{2}(m_u+m_d): \quad \frac{m_{\rm PS}^2}{f_\pi^2} \to \frac{m_\pi^2}{f_\pi^2}, \quad m_s: \quad \frac{m_{\rm PS}^2}{f_\pi^2} \to \frac{m_{\rm K}^2}{f_\pi^2}$$

Hadronic renormalisation scheme

- Hadronic input quantities fix the values of the bare coupling and quark masses

Example:

Parameter	Quantity
g_0	f_π
$\frac{1}{2}(m_u + m_d)$	m_π
m_s	m_{K}
m_c	$m_{\mathrm{D_s}}$
m_b	$m_{\mathrm{B_s}}$

- Except for input quantities, all other observables are predictions

3. Algorithms and Machines

The Hybrid Monte Carlo Algorithm

[Duane, Kennedy, Pendleton, Roweth, PLB 195 (1987) 216]

- "Hybrid": Molecular Dynamics + Metropolis accept/reject

- QCD with $N_\mathrm{f} = 2$ flavours of Wilson fermions:

$$D_\mathrm{w} = \tfrac{1}{2}\big(\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \nabla_\mu^*\nabla_\mu\big) + m_0 \qquad (= D_\mathrm{lat} + m_0)$$

$$Z = \int D[U]\,\det\left(D_\mathrm{w}^2\right)\,\mathrm{e}^{-S_\mathrm{G}[U]}$$

$$\det\left(D_\mathrm{w}^2\right) = \frac{1}{\det(D_\mathrm{w}^\dagger D_\mathrm{w})^{-1}} = \int D[\phi]\exp\left\{-\big(\phi, (D_\mathrm{w}^\dagger D_\mathrm{w})^{-1}\phi\big)\right\}$$

- Pseudo-fermion action:

$$S_\mathrm{pf}[U,\phi] = \big(\phi, (D_\mathrm{w}^\dagger D_\mathrm{w})^{-1}\phi\big) = \sum_x \left\|\left(D_\mathrm{w}^{-1}\phi\right)(x)\right\|^2$$

Molecular dynamics

- Introduce "fictitious" time variable t — "simulation time"
- Dynamical variable: $\{U_\mu(x)\}_t$
- Conjugate "momentum": $\{\Pi_\mu(x)\}_t$
- Hamiltonian: $\mathcal{H} = \frac{1}{2}(\Pi, \Pi) + S_{\mathrm{G}}[U] + S_{\mathrm{pf}}[U, \phi]$
- Equations of motion:

$$\frac{d}{dt} U_\mu(x) = \Pi_\mu(x) U_\mu(x)$$

$$\frac{d}{dt} \Pi_\mu(x) = -F_{\mathrm{G};\mu}(x) - F_{\mathrm{pf};\mu}(x)$$

$\longrightarrow$ evolve gauge field along trajectory on group manifold

Metropolis accept/reject step

- Numerical integration of e.o.m. in N steps:

$$\epsilon = \tau/N, \qquad \tau: \text{ trajectory length}$$

- Energy is not conserved:

$$\Delta\mathcal{H} = \mathcal{H}_{t_0+\tau} - \mathcal{H}_{t_0} \neq 0$$

$\longrightarrow$ Accept new configuration $\{U_\mu(x)\}_{t_0+\tau}$ with probability

$$\mathcal{P}[U_{t_0} \rightarrow U_{t_0+\tau}] = \min\left(1, \mathrm{e}^{-\Delta\mathcal{H}}\right) \quad \Rightarrow \quad \left\langle \mathrm{e}^{-\Delta\mathcal{H}} \right\rangle = 1$$

- global accept/reject step

Limitations

- Evaluation of $F_{\mathrm{pf};\mu}$ requires knowledge of D_{w}^{-1}

  $\longrightarrow$ solution of a linear system: $D_{\mathrm{w}}\psi = \eta$

- Efficiency depends on condition number of D_{w}

- Magnitude of $F_{\mathrm{pf};\mu}$ proportional to condition number

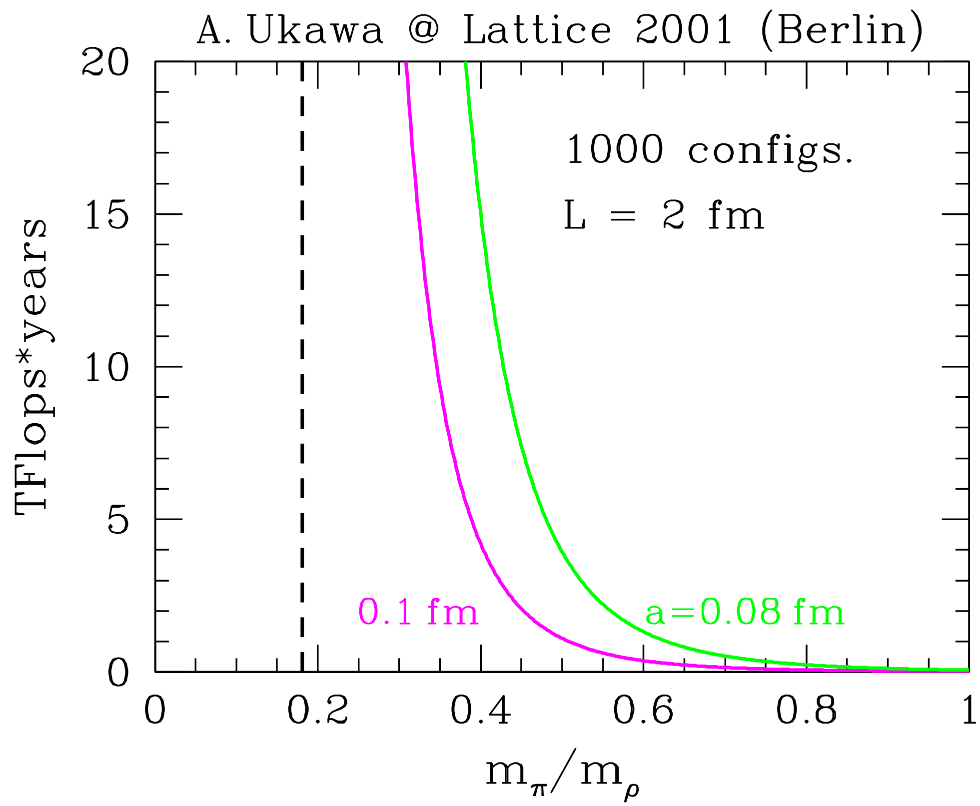
- Recall that $D_{\mathrm{w}} = D_{\mathrm{w}}^{(0)} + m_0$

  $\longrightarrow$ condition number increases as m_0 is tuned towards physical values of light quark masses

  $\longrightarrow$ Solution of linear system becomes ineffcient

  $\longrightarrow$ Must decrease step size ϵ in order to maintain reasonable acceptance rate

- **Berlin 2001:** panel discussion on cost of dynamical fermion simulations
- Estimate cost to generate 1000 independent configurations (Wilson quarks)



- Simulations with Wilson quarks not practical for $m_{\rm sea} < m_s/2$ and $a < 0.1\,{\rm fm}$

$\longrightarrow$ Impossible to reach domain of "realistic" pion masses?

Algorithmic improvements

- Domain decomposition
  [Lüscher 2003 – 05]
- Mass precoditioning + multiple integration timescales
  [Hasenbusch, Jansen 2001; Peardon & Sexton 2002; Urbach et al. 2005]
- Rational Hybrid Monte Carlo
  [Clark & Kennedy 2006]
- Deflation acceleration
  [Lüscher 2007, Morgan & Wilcox 2007]
- Low-mode reweighting
  [Jansen et al. 2007, A. Hasenfratz et al. 2008; Palombi & Lüscher 2008]

Domain decomposition methods for QCD

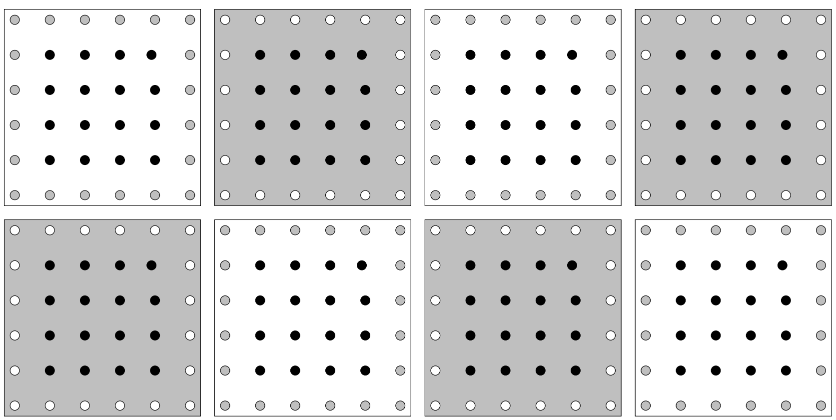
[M. Lüscher 2003–05]

Hermann Schwarz 1870

Solution of Dirichlet problem in complicated domains

Solve Laplace equation alternately in overlapping sub-domains

$\longrightarrow$ Domain Decomposition



$\longrightarrow$ Exact factorisation of quark determinant:

$$\det(D_{\mathrm{w}}) = \prod_{\text{blocks } \Lambda} \det D_\Lambda \times \det R$$

Inter-block interaction: R

- D_Λ : Wilson-Dirac operator with Dirichlet boundary conditions

- Pseudofermion action:

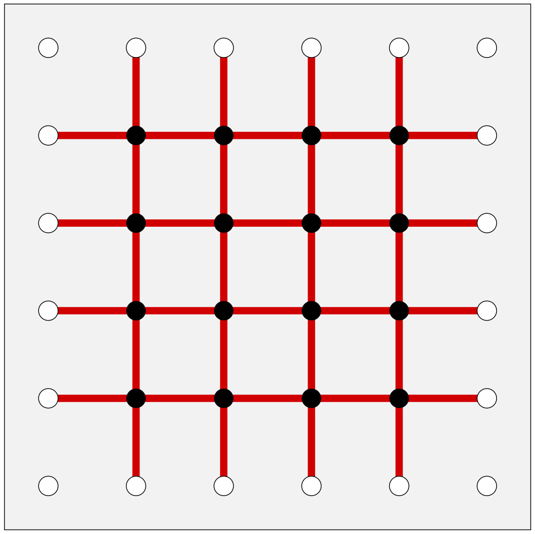
$$S_{\rm pf} = \sum_{\text{blocks } \Lambda} \left\| D_\Lambda^{-1}\phi_\Lambda \right\|^2 + \left\| R^{-1}\chi \right\|^2$$

- Block size l $\sim$ IR cutoff

$$l < 0.5\,\text{fm} \quad \Rightarrow \quad q \geq \pi/l > 1\,\text{GeV}$$

$\longrightarrow$ theory weakly coupled

$\longrightarrow$ easy to simulate at all quark masses

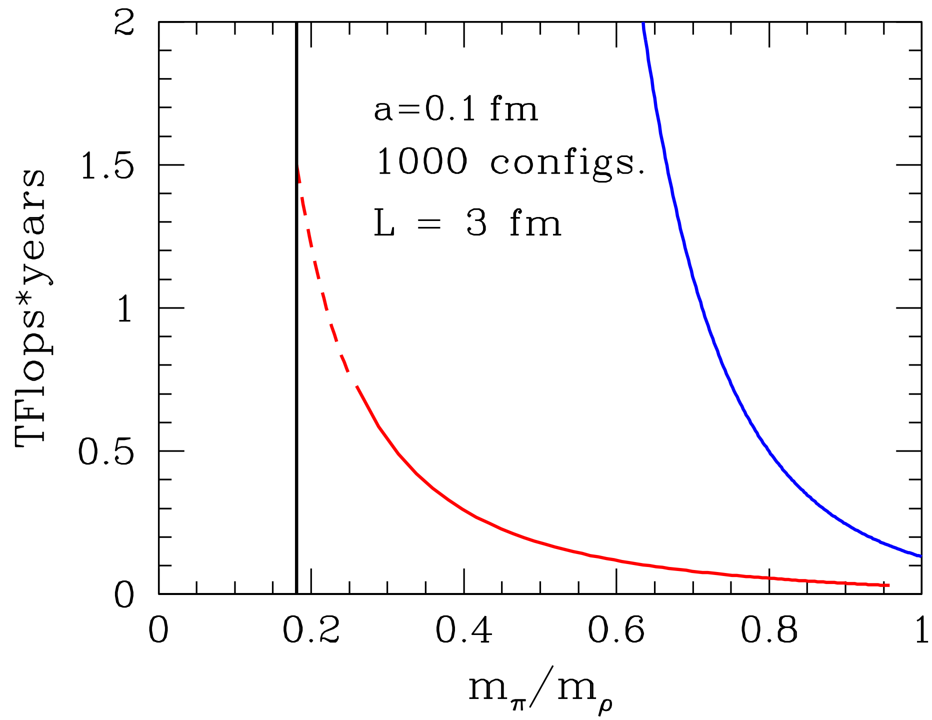


- Mode separation:

$$\det(D_{\rm w}) = \prod_{\text{blocks } \Lambda} \det D_\Lambda \underbrace{\times \det R}_{\text{long range}}$$

The Berlin Wall revisited

- DD-HMC algorithm scales slowly with quark mass
- blocks are mapped onto nodes of parallel computer
- most CPU time spent on sub-domain
  $\longrightarrow$ reduced communication overhead



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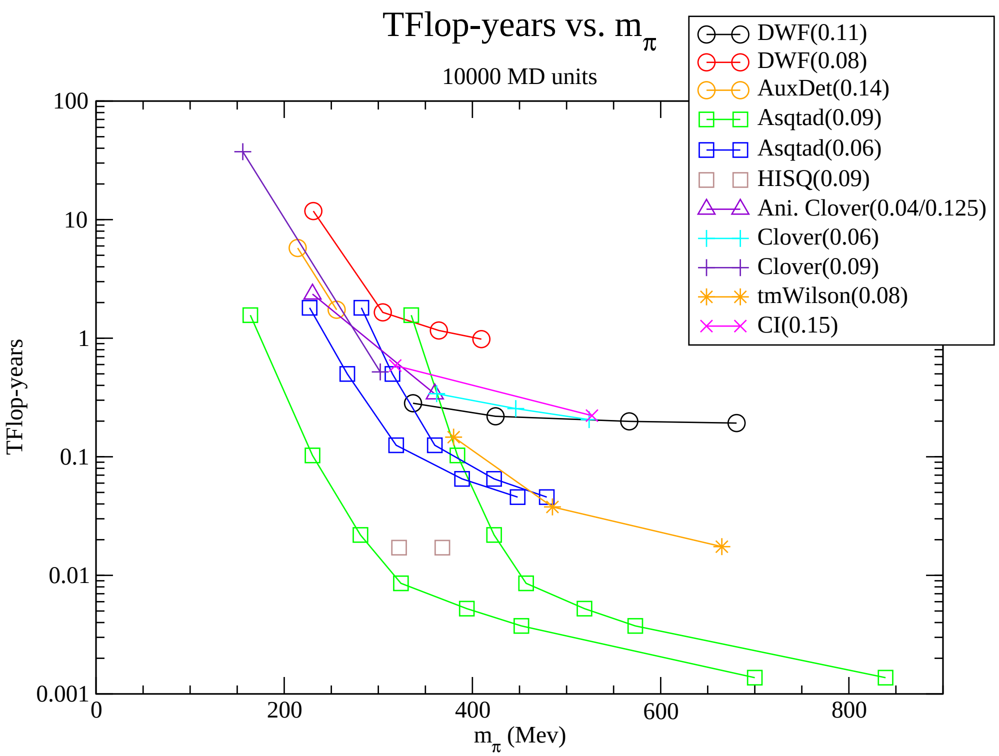

[Ch Jung Lattice 2009]

Computing platforms in Lattice QCD

- Commercial supercomputers:

  BlueGene/P(L), SGI Altix, Hitachi SR8000, NEC Sx6, Fujitsu VPP700,. . .

- Custom made machines:

QCDOC	~ 10 TFlop/s	2004	CU/UKQCD/Riken/IBM
apeNEXT	~ 10 TFlop/s	2005	INFN/DESY/Paris-Sud
QPACE	~ 50 TFlop/s	2009	Regensburg/IBM

- PC clusters + fast network:

  – Mass-produced components $\longrightarrow$ cheap

  – Typically larger latencies, smaller bandwidths

  $\Rightarrow$ scalability not as good as for custom made machines

- Graphics processors:

  – Driven by the video games market; enormous peak speeds

  – Significant programming effort; poor parallelisation

Graphics processors

[Schömer, Walk, HW]

- Video games market drives development of powerful processors
- Can be exploited for scientific purposes
- Implementation of Wilson-Dirac operator
- Compare CPU speeds without parallelisation (32-bit precision)

Other hardware platforms: graphics processors

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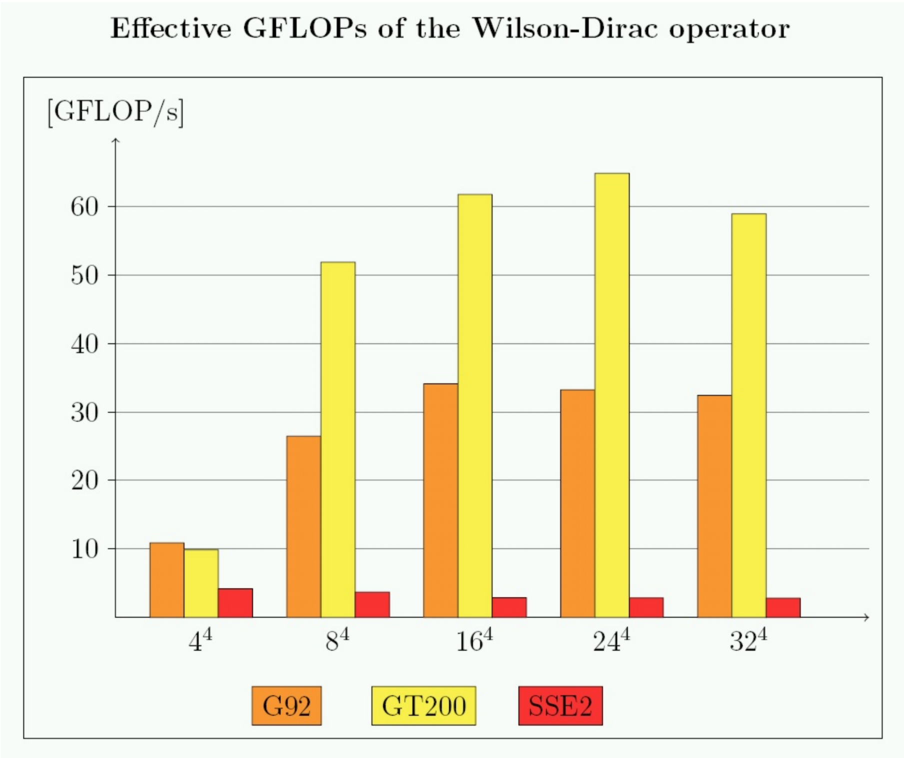

- GPUs competitive in terms of raw compute speed
- Memory- and communication bandwidths & latencies limit scalability

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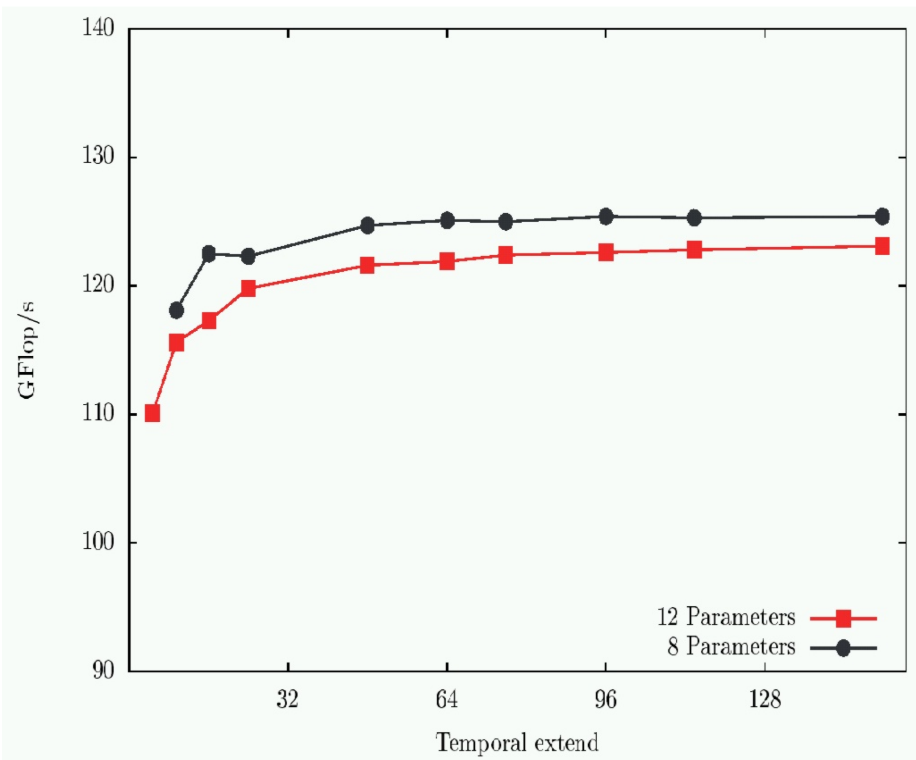

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Summary - part I

- Lattice formulation: "ab initio" method for QCD at low energies

- Conceptually well founded;

  truly non-perturbative procedure which respects gauge invariance

- Recent progress:

  - chiral symmetry + lattice regularisation
  - faster fermion algorithms
  - faster machines
  - refined methods

Lattice — Recent Results

Current status

Simulations with light dynamical quarks have become routine

Distinguish different classes of observables:

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Simulations with light dynamical quarks have become routine

Distinguish different classes of observables:

Class I: "Precision observables"; overall errors of few $- 10\,\%$

- spectrum of lowest-lying hadrons
- light quark masses
- $f_{\rm K}/f_\pi$, form factors for $K_{\ell 3}$-decays
- some heavy-light decay constants and form factors;
- B-parameter $B_{\rm K}$ (?)
- . . .

$\longrightarrow$ Focus on further reduction of systematic errors

Class II: Semi-quantitative results; overall error of $\approx 10-50\%$

- nucleon form factors and structure functions
- excitation spectrum; nucleon resonances
- $K \to \pi\pi$ and the $\Delta I = 1/2$-rule
- hadronic vacuum polarisation contribution to $(g-2)_\mu$
- . . .

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- . . .

Class III: Exploratory calculations; no quantifyable error

- critical endpoint of deconfining phase transition at finite density
- $K \to \pi\pi$ and the value of ϵ'/ϵ
- SUSY on the lattice
- . . .

4. Hadron spectroscopy

Spectrum calculations: basic ingredients

- Choose bare parameters: coupling $\beta = 6/g_0^2$ and sea quark mass(es) $m^{\rm sea}$

- Generate ensemble of gauge configurations

- Compute correlation functions: $\sum_{\vec{x}} \left\langle \phi_{\rm had}(x)\phi^\dagger_{\rm had}(0) \right\rangle \sim {\rm e}^{-m_{\rm had}x_0}$

- $\phi_{\rm had}(x)$: interpolating operator for given hadron:

$$
\begin{aligned}
K\text{-meson} &: \quad \phi_{\rm K} = s\,\gamma_5\,\overline{u}, \quad s\,\gamma_0\gamma_5\,\overline{u} \\
\text{nucleon} &: \quad \phi_{\rm N} = \varepsilon_{abc}\big(u^a\,C\gamma_5\,d^b\big)u^c \\
\Delta &: \quad \phi_{\Delta} = \varepsilon_{abc}\big(u^a\,C\gamma_\mu\,d^b\big)u^c
\end{aligned}
$$

- Interpolating operators project on all states with the same quantum numbers:

$$\sum_{\vec{x}} \mathrm{e}^{i\vec{p}\cdot\vec{x}} \left\langle \phi_{\mathrm{had}}(x)\phi^{\dagger}_{\mathrm{had}}(0) \right\rangle = \sum_{n} w_n(\vec{p})\, \mathrm{e}^{-E_n(\vec{p})x_0}$$

$$w_n(\vec{p}) \equiv \frac{|\langle 0|\phi_{\mathrm{had}}|n\rangle|^2}{2E_n(\vec{p})} :$$ spectral weight of n^{th} state

- Magnitude of $w_n(\vec{p})$ depends on particular choice of ϕ_{had}

- Ground state dominates at large Euclidean times

- Excited states are sub-leading contributions

Challenges for spectrum calculations

- Isolate sub-leading contributions to correlation functions

  $\longrightarrow$ construct interpolating operators which maximise spectral weight $w_n(\vec{p})$ for a given state

- Capture properties of states with radial & orbital excitations

- Distinguish resonances from multi-hadron states

- Keep statistical noise small — especially near chiral regime

- Techniques:
  - Smearing
  - Variational principle
  - Stochastic sources
  - Anisotropic lattices

Smearing

- Hadrons are extended objects; $\phi_{\rm N}(x)=\varepsilon_{abc}\big(u^a\,C\gamma_5\,d^b\big)u^c$ is point-like

  $\longrightarrow$ expect small spectral weight associated with $\phi_{\rm N}$

- Enhance projection of interpolating operators for hadrons by "smearing" the quark fields:

$$\widetilde{\psi}(\vec{x},t)=\sum_{\vec{y}}F(\vec{x},\vec{y};U)\psi(\vec{y},t)$$

  $F(\vec{x},\vec{y};U)$: smearing function (gauge invariant)

- Gaussian smearing: $F(\vec{x},\vec{y};U)=(1+\kappa_S\,H)^{n_\sigma}(\vec{x},\vec{y};U)$

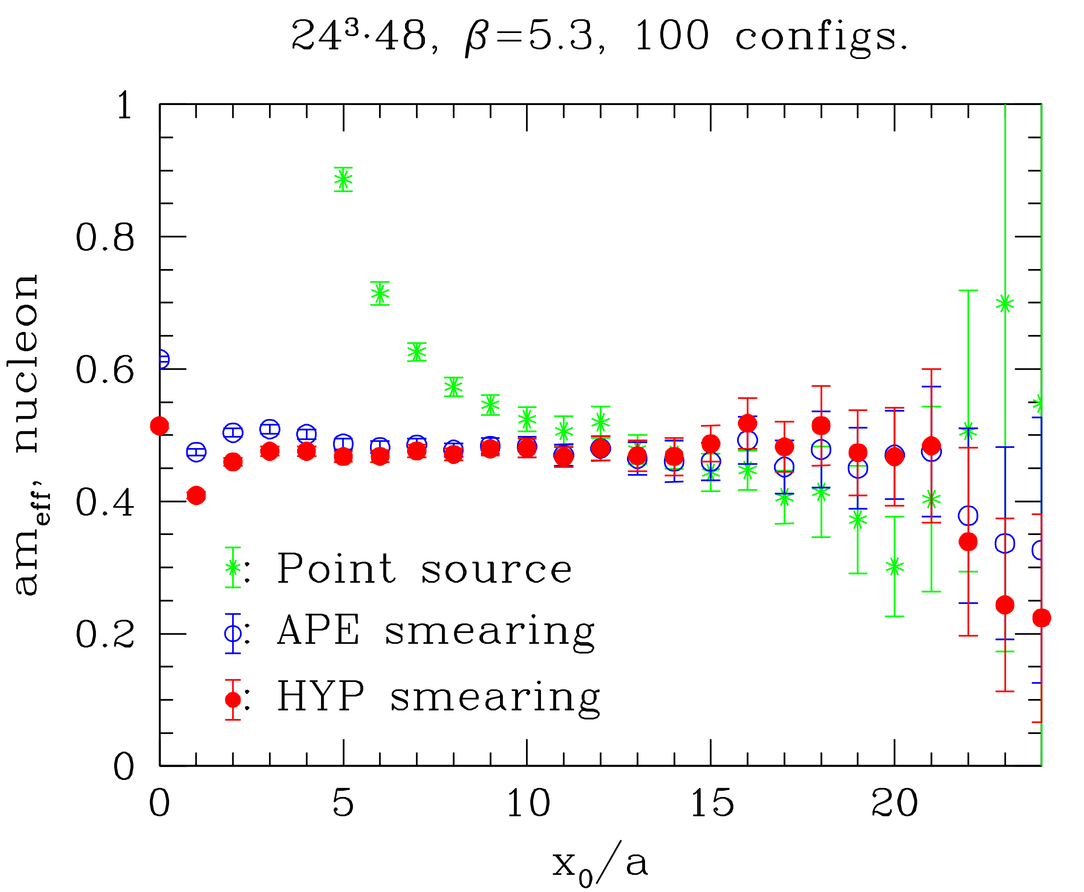
  H : covariant Laplacian in 3D

- Variants: Jacobi smearing,. . .

- Further improvement by replacing spatial links $U_j(x)$ by smeared ones, e.g. "APE", "HYP", "stout"-smearing

- Effective mass plot:

$$C(t) \sim \mathrm{e}^{-mt} \quad \Rightarrow \quad m_{\text{eff}}(t) = \ln C(t)/C(t+a)$$



[Capitani, Della Morte, Jüttner, Knippschild, H.W., in preparation]

Systematic effects

Lattice artefacts:

$$\left\langle\frac{m_N}{f_\pi}\right\rangle^{\text{lat}} = \left\langle\frac{m_N}{f_\pi}\right\rangle^{\text{cont}} + O(a^p), \quad p \geq 1$$

$\longrightarrow$ requires extrapolation to continuum limit, $a \to 0$

Finite volume effects:

- Mass estimates distorted by finite box size
- Rule of thumb: $L \approx 2.5-3\,\text{fm}$ and $m_\pi L > 3-4$ sufficient for many purposes (?)

Unphysical quark masses:

- Rely on chiral extrapolations to physical values of m_u, m_d
- Use Chiral Perturbation Theory (ChPT) to guide extrapolations
- Chiral corrections reliably described by ChPT?

Ground state mesons & baryons

- Calculation of lowest-lying octet & decuplet baryons (strange and non-strange)

  $\longrightarrow$ Benchmark of lattice QCD

- Variety of different discretisations; continuum extrapolations

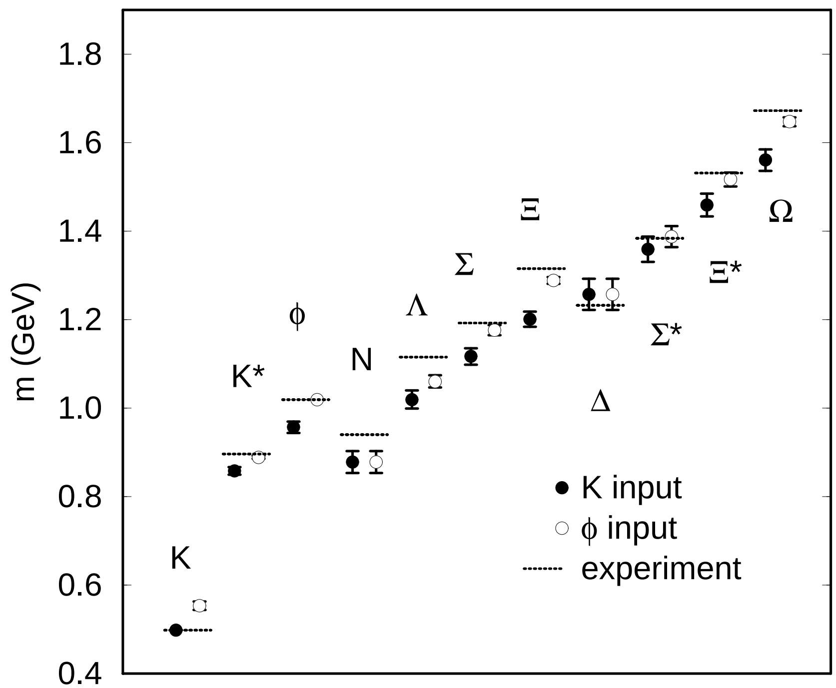
  $\longrightarrow$ Check on lattice artefacts

- Focus on control over systematics:

  $\longrightarrow$ chiral extrapolations, finite-size effects, lattice artefacts

CP-PACS Collaboration (Aoki et al.) Phys Rev D67 (2003) 034503

- Wilson fermions; 4 values of a $\Rightarrow$ continuum extrapolation
- Lattice scale set by m_ρ



- Set strange quark mass by

$$\frac{m_{\mathrm{PS}}^2}{m_\rho^2} \stackrel{!}{=} \frac{m_{\mathrm{K}}^2}{m_\rho^2} \quad (\text{“K-input”}) \quad \text{or}$$

$$\frac{m_{\mathrm{V}}}{m_\rho} \stackrel{!}{=} \frac{m_\phi}{m_\rho} \quad (\text{“}\phi\text{-input”})$$

- Experimentally observed spectrum reproduced at the level of $10 - 15\%$
- Small but significant deviations

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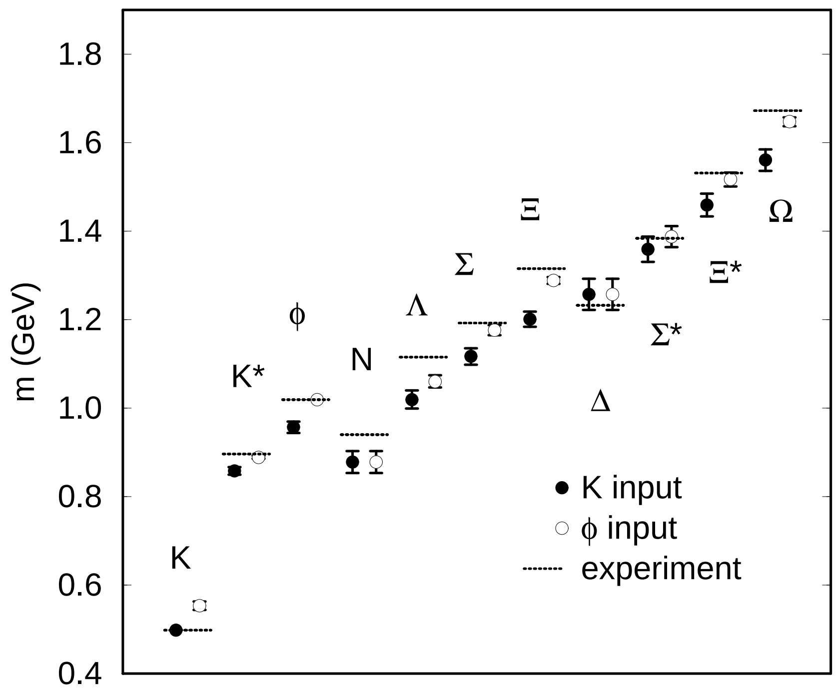

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- Quenched approximation

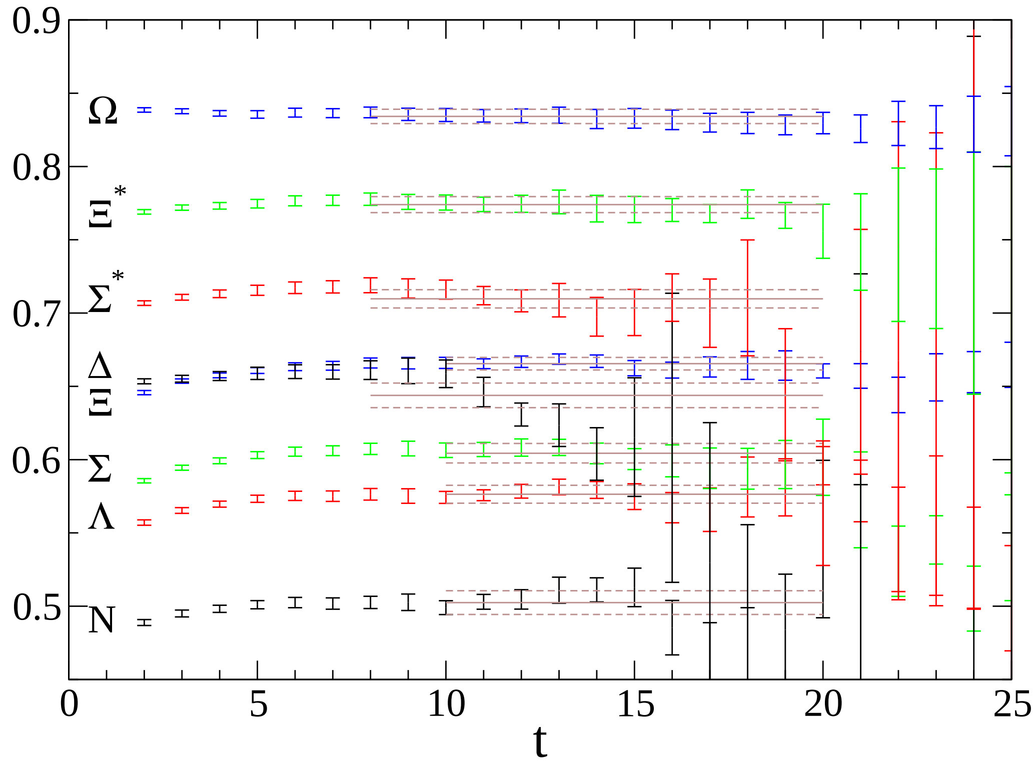
PACS-CS Collaboration (Aoki et al.) Phys Rev D79 (2009) 034503

- $N_\mathrm{f} = 2 + 1$, Clover fermions & Iwasaki gauge action
- Single lattice spacing: $a = 0.09\,\mathrm{fm}$ (set by m_Ω)
- Pion masses: $m_\pi = 156,\ 296,\ 412,\ 571,\ 702\,\mathrm{MeV}$
- Volume: $32^3 \cdot 64,\quad L \simeq 2.9\,\mathrm{fm},\quad m_\pi^{\mathsf{min}} L = 2.3$

$\longrightarrow$ Almost at physical m_π but likely to suffer from finite-size effects

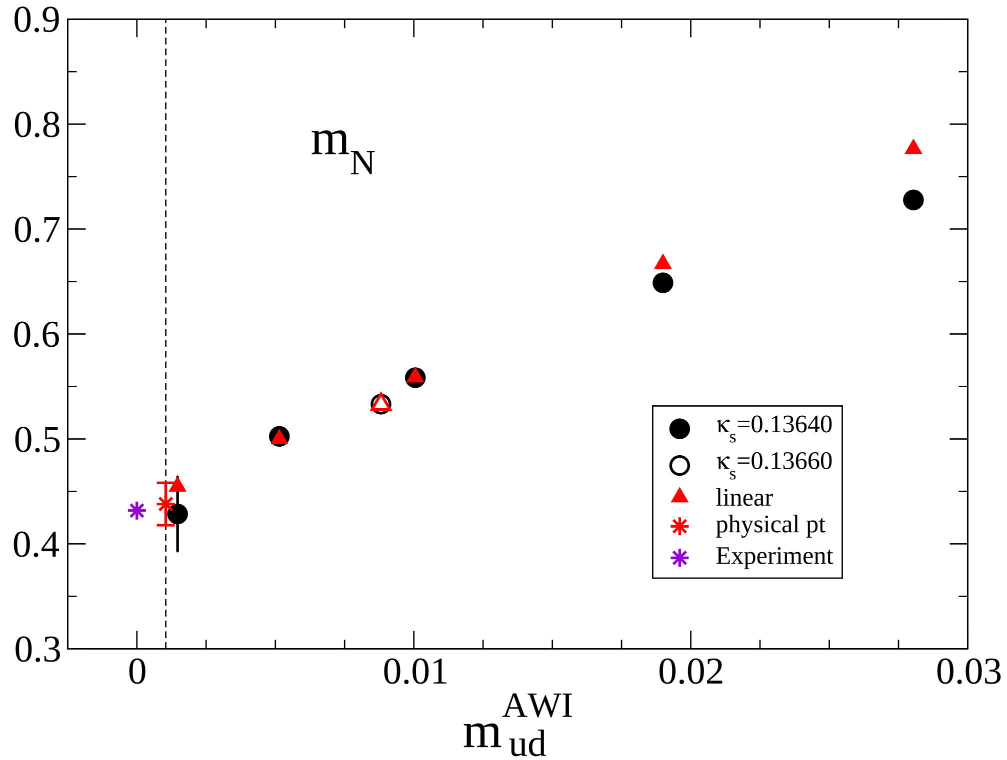
PACS-CS Collaboration (Aoki et al.) Phys Rev D79 (2009) 034503

- Effective mass plots at fixed pion mass



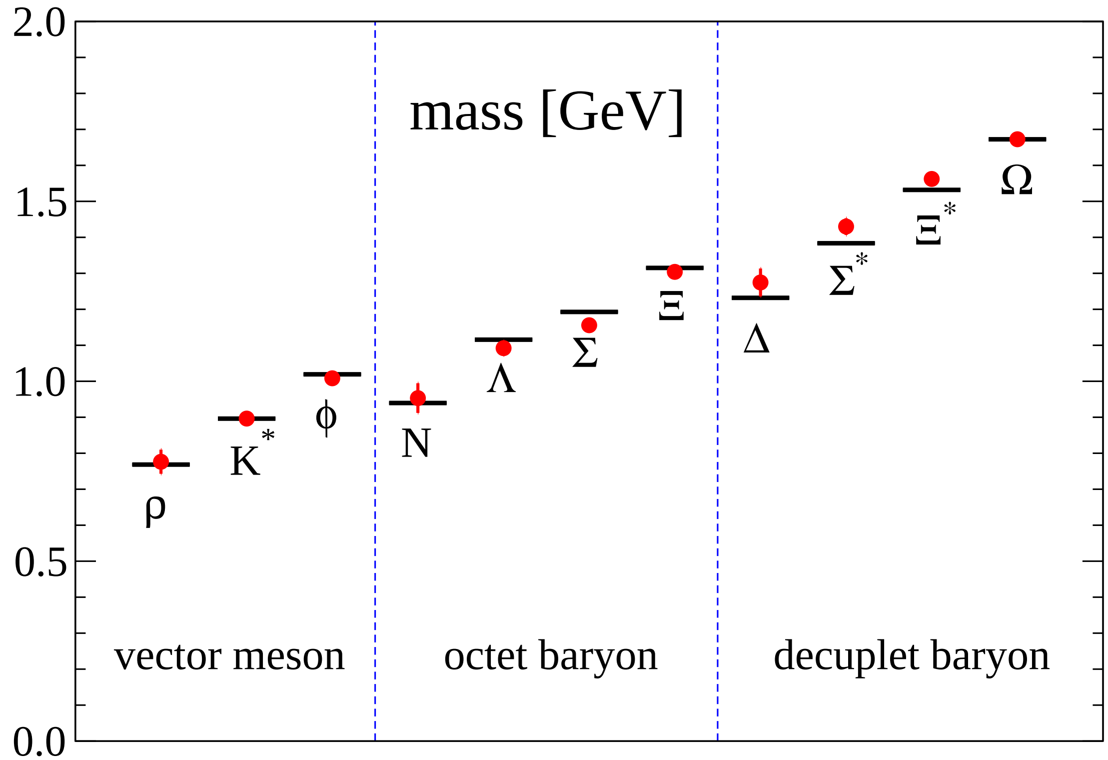
PACS-CS Collaboration (Aoki et al.) Phys Rev D79 (2009) 034503

- Chiral extrapolations of the nucleon



PACS-CS Collaboration (Aoki et al.) Phys Rev D79 (2009) 034503

- Result for lowest-lying hadrons at fixed lattice spacing $a \approx 0.09\,\mathrm{fm}$



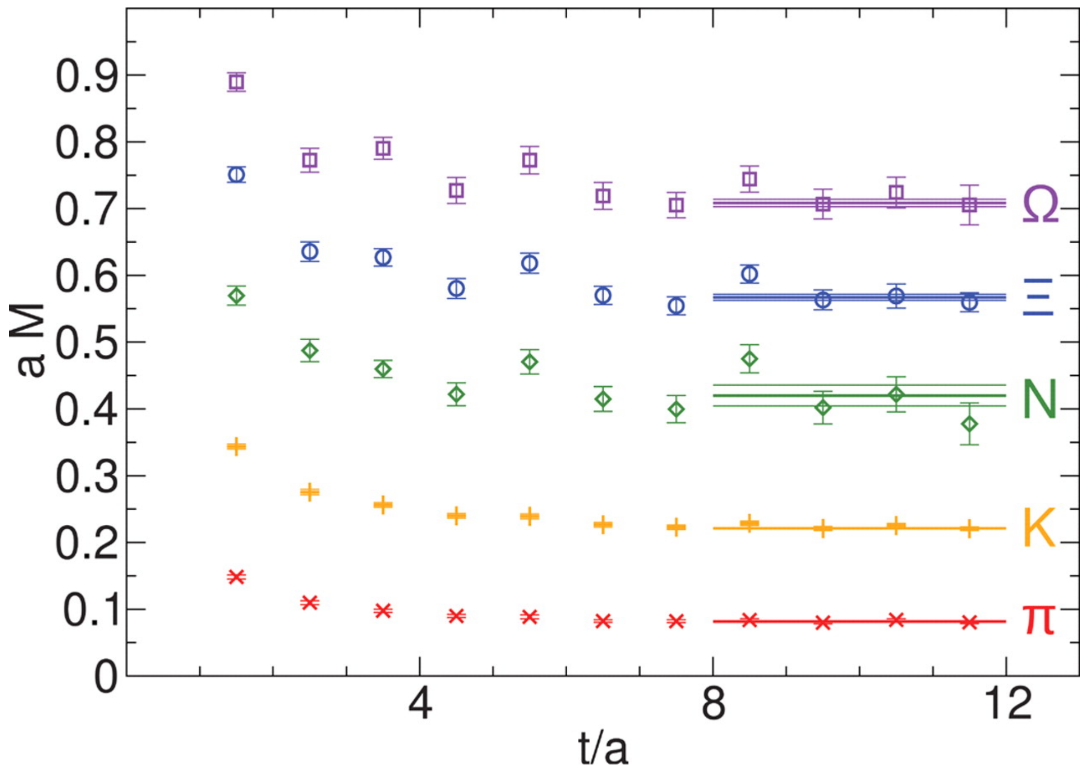
- Good agreement with experimental spectrum;
  small discrepancies possibly due to lattice artefacts

BMW Collaboration (Dürr et al.) Science 322 (2008) 1224

- $N_\mathrm{f} = 2 + 1$, smeared Clover fermions & tree-level improved Symanzik gauge action
- 3 lattice spacings: $a \approx 0.125,\ 0.085,\ 0.065\,\mathrm{fm}$ (set by m_Ξ)
- Pion masses: $m_\pi \gtrsim 190\,\mathrm{MeV}$
- Volume: $m_\pi^{\mathsf{min}} L \gtrsim 4$ throughout; largest lattice $L/a = 48$
- Introduce non-localities by too much smearing in fermion action?

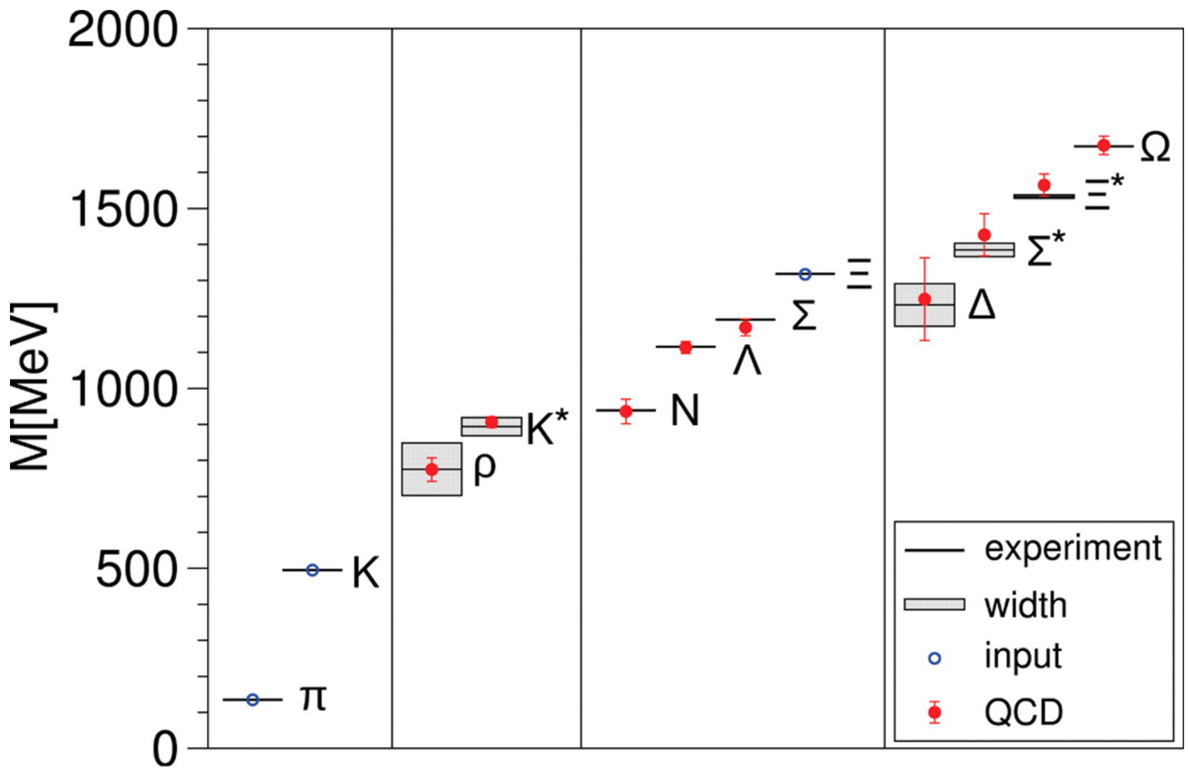
BMW Collaboration (Dürr et al.) Science 322 (2008) 1224

- Effective mass plots for $m_\pi \approx 190\,\mathrm{MeV},\ a \approx 0.085\,\mathrm{fm},\ L/a = 48$



BMW Collaboration (Dürr et al.) Science 322 (2008) 1224

- Lowest-lying hadrons in continuum limit



- Experimentally observed spectrum well reproduced

Current status

- Significant progress in understanding the masses of lowest-lying mesons & baryons from first principles
- Systematics such discretisation errors, chiral extrapolations controlled; towards precision determinations
- Further progress needed to understand resonances
- QCD has been confirmed as the theory of the strong interaction

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- Independent confirmation for different discretisations still required

5. Kaon physics

- Kaon decays and $K^0 - \bar{K}^0$ mixing: input for flavour physics
- Use experimental and theoretical input to determine CKM matrix elements:

$$V_{\rm CKM} = \begin{pmatrix} V_{\rm ud} & V_{\rm us} & V_{\rm ub} \\ V_{\rm cd} & V_{\rm cs} & V_{\rm cb} \\ V_{\rm td} & V_{\rm ts} & V_{\rm tb} \end{pmatrix} \approx \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\bar{\rho}-i\bar{\eta}) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\bar{\rho}-i\bar{\eta}) & -A\lambda^2 & 1 \end{pmatrix}$$

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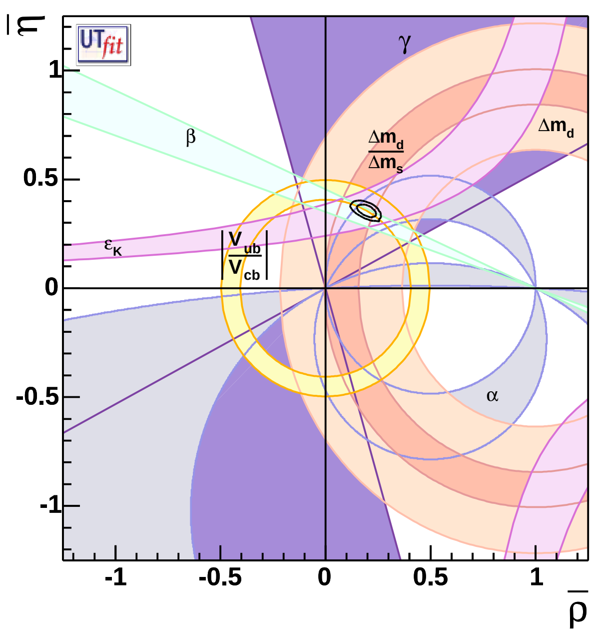

- Unitarity triangle:

$$V_{\rm ud}V_{\rm ub}^* + V_{\rm cd}V_{\rm cb}^* + V_{\rm td}V_{\rm tb}^* = 0,$$

- Indirect CP-violation in kaon sector constrains apex:

$$\epsilon_{\rm K} \propto \widehat{B}_{\rm K}$$

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- First-row unitarity:

$$|V_{\rm ud}|^2 + |V_{\rm us}|^2 + |V_{\rm ub}|^2 = 1$$

- $K_{\ell 3}$-decays: $\Gamma(K \to \pi\ell\nu_\ell) \propto \dfrac{G_{\rm F}^2 m_{\rm K}^5}{192\pi^3} |V_{us}|^2 \left|f_+^{K\pi}(0)\right|^2$

- Precision of unitarity tests limited by uncertainty in $f_+^{K\pi}(0)$

$K^0 - \overline{K}^0$ mixing and the kaon B-parameter

- Effective weak Hamiltonian describes $\Delta S = 2$ transitions:

$$\mathcal{H}_{\rm eff}^{\Delta S=2} = \frac{G_{\rm F}^2 M_{\rm W}^2}{16\pi^2}\,\mathcal{F}^0\, O^{\Delta S=2} + \text{h.c.}$$

$$O^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma_\mu(1-\gamma_5)d] = O_{\rm VV+AA}^{\Delta S=2} - O_{\rm VA+AV}^{\Delta S=2}$$

$$\mathcal{F}^0 = \lambda_{\rm c}^2 S_0(x_{\rm c})\eta_1 + \lambda_{\rm t}^2 S_0(x_{\rm t})\eta_2 + 2\lambda_{\rm c}\lambda_{\rm t} S_0(x_{\rm c}, x_{\rm t})\eta_3$$

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- Relation to $\epsilon_{\rm K}$:

$$\epsilon_{\rm K} = {\rm e}^{i\phi_\epsilon}\sin\phi_\epsilon \left(\frac{{\rm Im}\,(\langle \bar{K}^0 | \mathcal{H}_{\rm eff}^{\Delta S=2} | K^0 \rangle)}{\Delta M_{\rm K}} + \frac{{\rm Im}\,A_0}{{\rm Re}\,A_0} \right)$$

- Matrix element of $O^{\Delta S=2}$ expressed in terms of the kaon B-parameter $B_{\rm K}$:

$$B_{\rm K}(\mu) = \frac{\left\langle \overline{K}^0 \left| O^{\Delta S=2}(\mu) \right| K^0 \right\rangle}{\frac{8}{3} f_{\rm K}^2 m_{\rm K}^2}$$

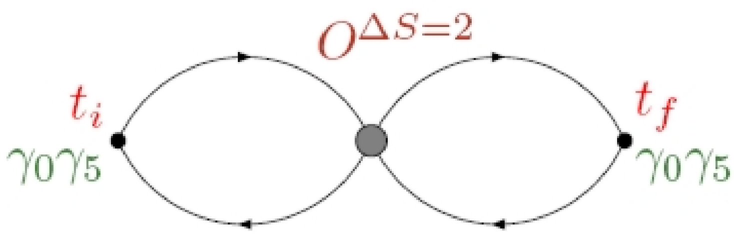
- Renormalisation group invariant (RGI) B-parameter:

$$\widehat{B}_{\rm K} = \left(\frac{\overline{g}(\mu)^2}{4\pi}\right)^{\gamma_0/2b_0} \exp\left\{\int_0^{\overline{g}(\mu)} dg \left(\frac{\gamma(g)}{\beta(g)} + \frac{\gamma_0}{\beta_0 g}\right)\right\} B_{\rm K}(\mu),$$

$$\simeq \left(\frac{\overline{g}(\mu)^2}{4\pi}\right)^{\gamma_0/2b_0} \left\{1 + \overline{g}(\mu)^2 \left[\frac{b_0\gamma_1 - b_1\gamma_0}{2b_0^2}\right]\right\} B_{\rm K}(\mu),$$

Lattice calculations

- Compute 3-point correlation functions: $\phi_K(x) = (s\gamma_0\gamma_5\overline{d})(x)$



$$\sum_{\vec{x}_i,\vec{x}_f} \left\langle \phi_K(x_f) O^{\Delta S=2}(0) \phi_K^\dagger(x_i) \right\rangle$$

$$\sim \mathrm{e}^{-m_K\,(T-t_i)}\,\mathrm{e}^{-m_K\,t_f}\,\frac{|\zeta|^2}{4m_K^2}\left\langle \overline{K}^0 \left| O^{\Delta S=2}(0) \right| K^0 \right\rangle$$

$m_K,\ \zeta = \left\langle 0 \left| s\gamma_0\gamma_5\overline{d} \right| K^0 \right\rangle$ known from 2-point function

- Compute suitable ratios of 3- and 2-point functions:

$$\frac{\displaystyle\sum_{\vec{x}_i,\vec{x}_f} \left\langle \phi_K(x_f) O^{\Delta S=2}(0) \phi_K^\dagger(x_i) \right\rangle}{\displaystyle\sum_{\vec{x}_i} \langle \phi_K(x_i)\phi_K^\dagger(0)\rangle \sum_{\vec{x}_f} \langle \phi_K(x_f)\phi_K^\dagger(0)\rangle} \propto B_K(\mu)$$

Renormalisation and mixing

- Explicit chiral symmetry breaking:

  $O_{\mathrm{VV+AA}}(\mu)$ mixes under renormalisation:

$$O^{\mathrm{R}}_{\mathrm{VV+AA}}(\mu) = Z(g_0, a\mu) \left\{ O^{\mathrm{bare}}_{\mathrm{VV+AA}} + \sum_{i=1}^{4} \Delta_i(g_0) O_i^{\mathrm{bare}} \right\}$$

  Wilson fermions: explicit chiral symmetry breaking: $\Delta_i \neq 0$

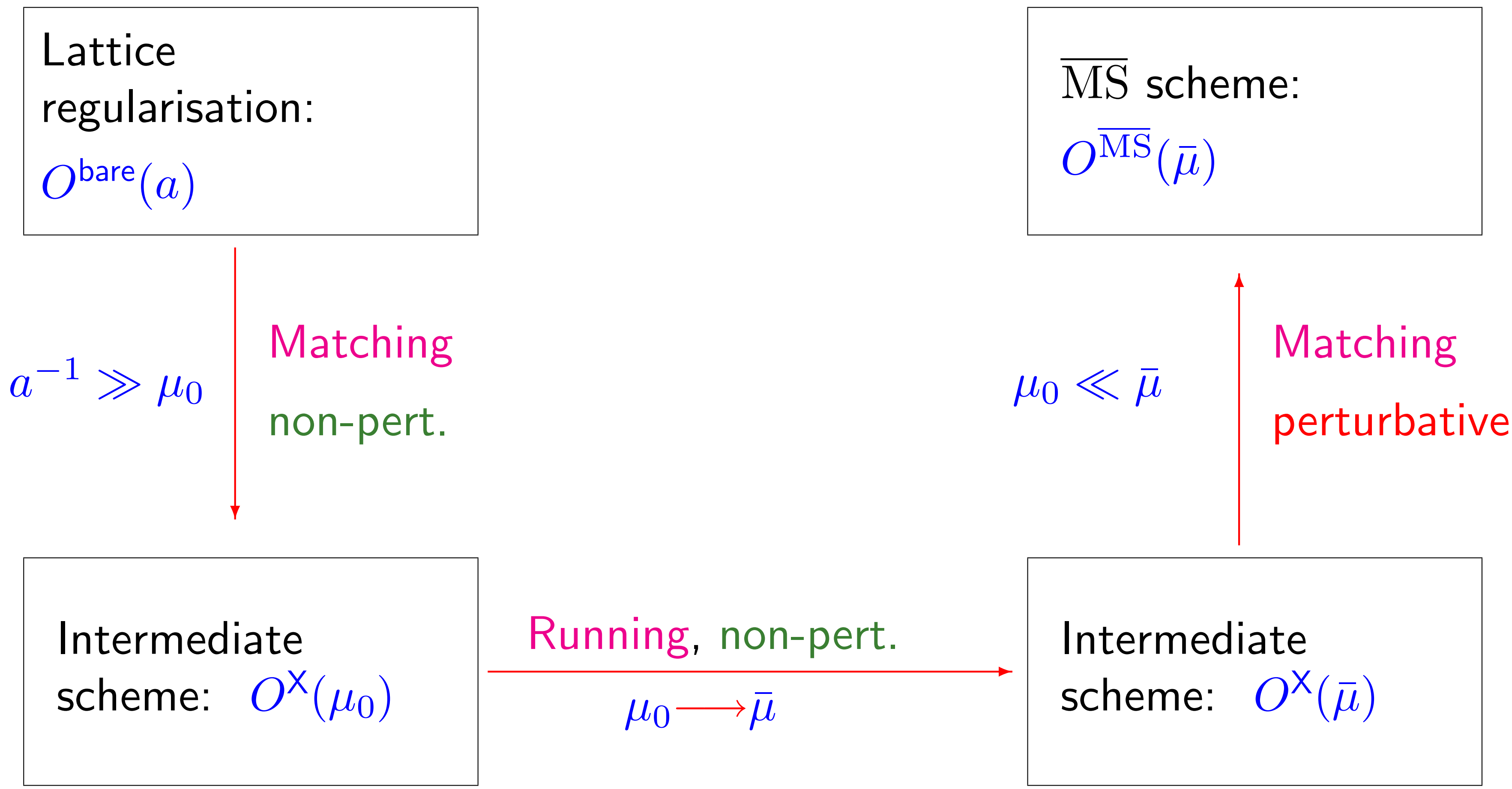
  Staggered fermions: Remnant chiral symmetry: $\Delta_i = 0$

  Domain Wall/Overlap: chiral symmetry preserved; expensive to simulate

  "Twisted mass" QCD: operator $O^{\mathrm{bare}}_{\mathrm{VV+AA}}$ can be "rotated" to $O^{\mathrm{bare}}_{\mathrm{VA+AV}}$

- $Z(g_0, a\mu)$ and $\Delta_i(g_0)$ can be computed non-perturbatively via intermediate renormalisation schemes

Intermediate renormalisation schemes



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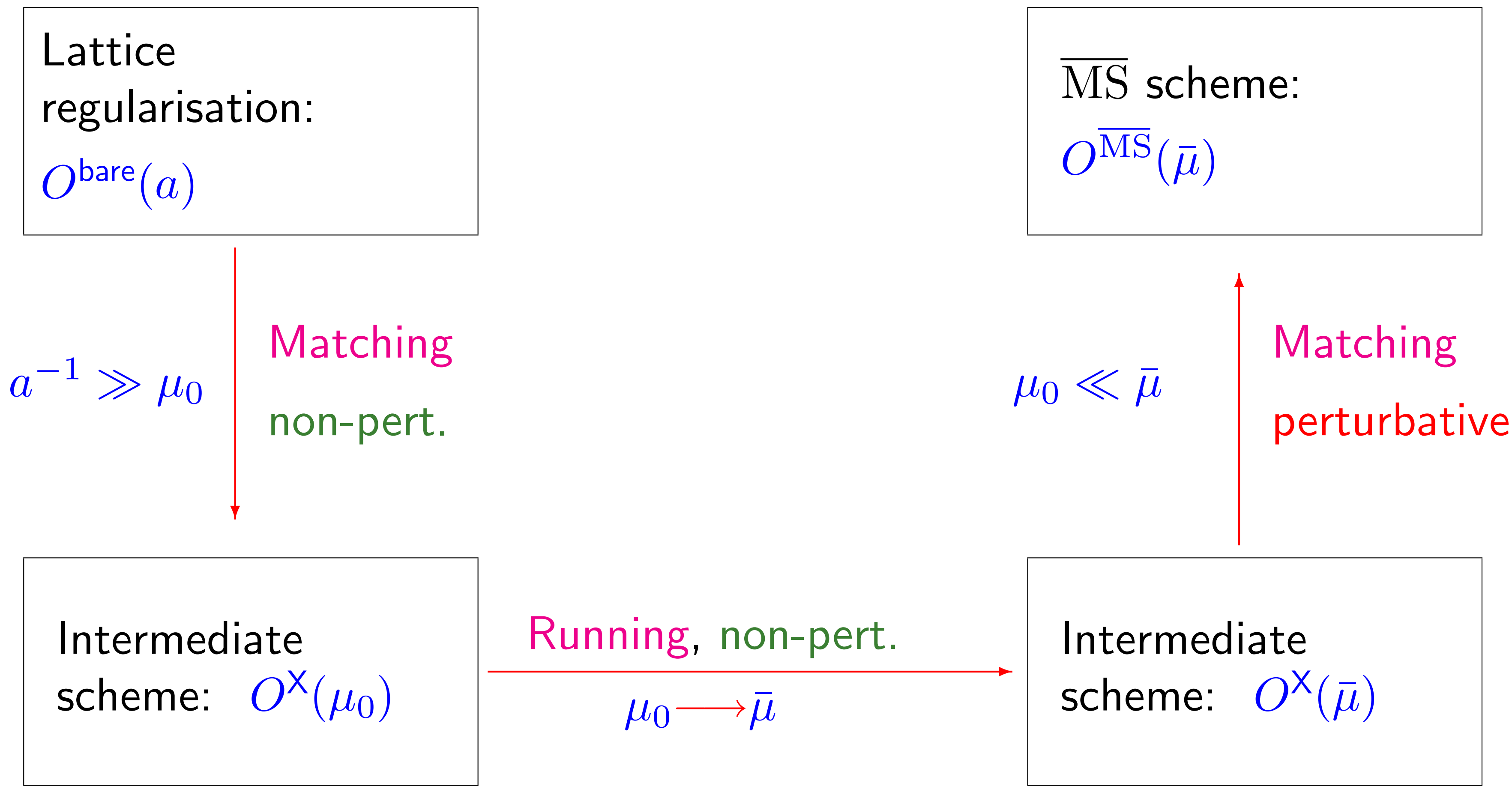
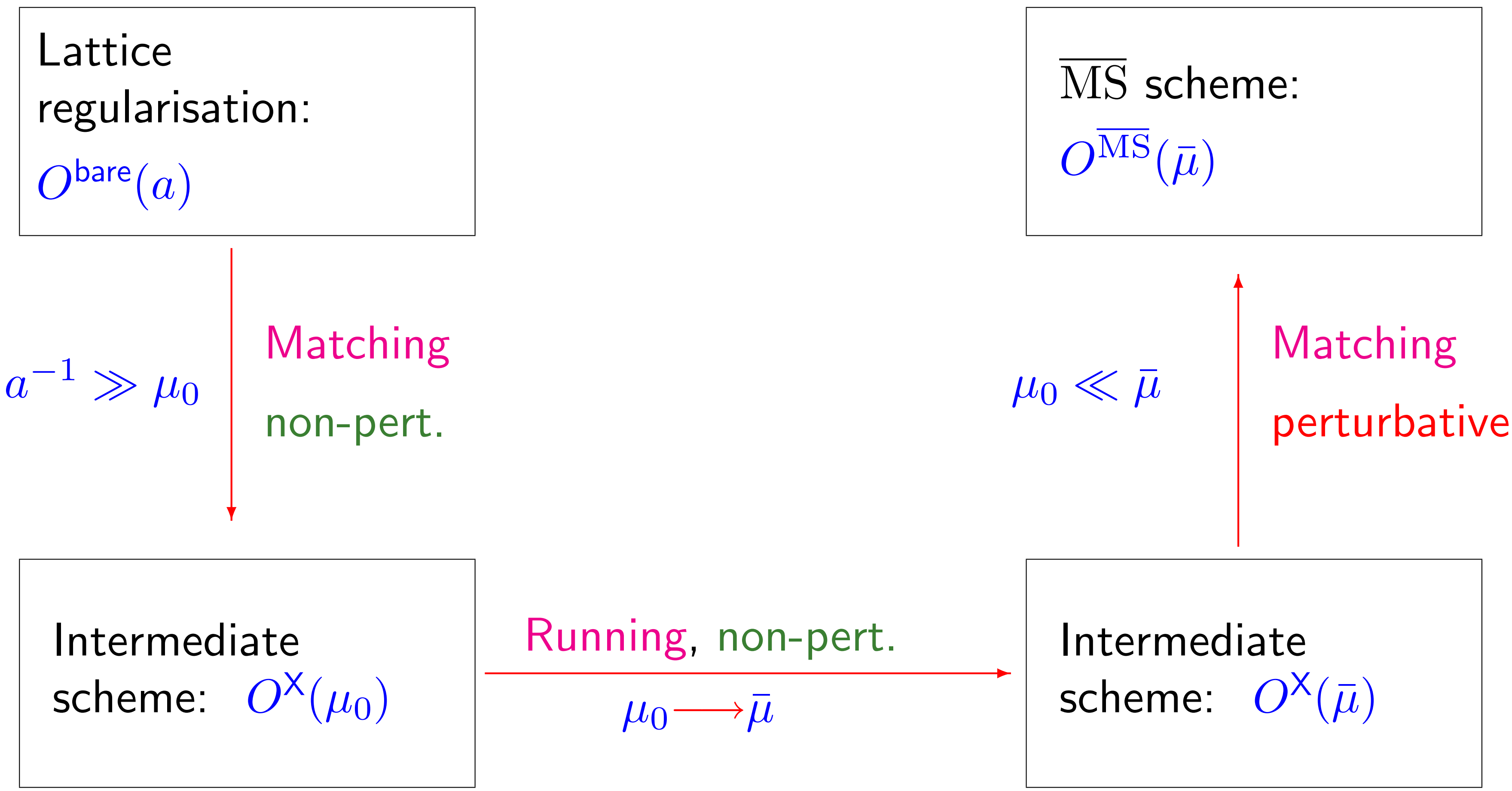


- **Examples:** Regularisation-independent momentum-subtraction (RI/MOM)
  [Martinelli, Pittori, Sachrajda, Testa, Vladikas, 1994]

Intermediate renormalisation schemes

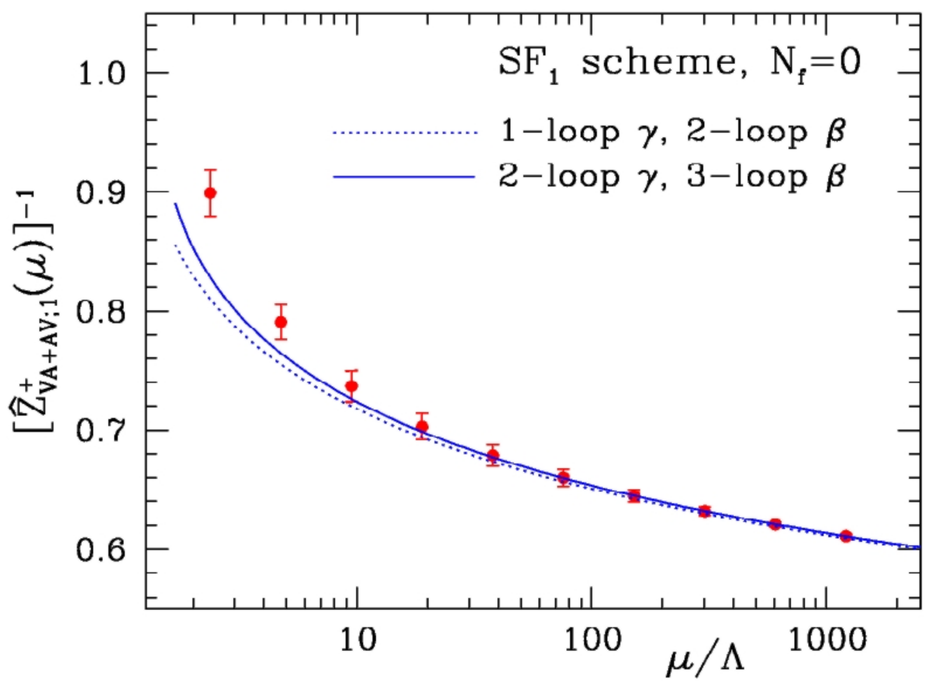


- **Examples:** Regularisation-independent momentum-subtraction (RI/MOM)
  [Martinelli, Pittori, Sachrajda, Testa, Vladikas, 1994]

  Schrödinger Functional (SF) *[Lüscher et al., 1992–]*

ALPHA Collaboration, Nucl Phys B749 (2006) 69, Nucl Phys B776 (2007) 258

- Quenched approximation
- Twisted mass fermions; 5 values of a $\Rightarrow$ continuum extrapolation
- Use Schrödinger functional technique for non-perturbative determination of the total renormalisation factor, $Z_{\mathrm{VA+AV}}$



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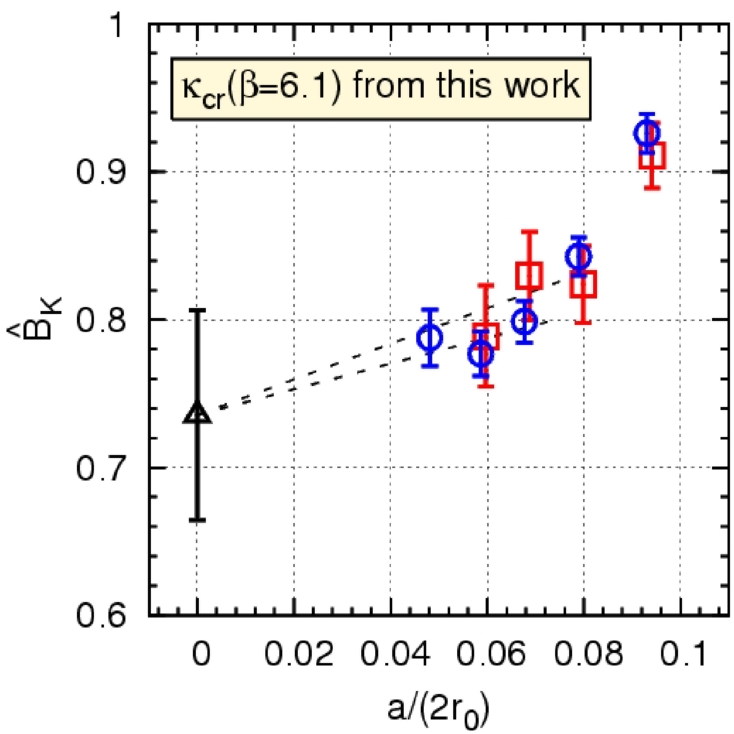


$$\widehat{B}_{\mathrm{K}} = 0.735(71),$$

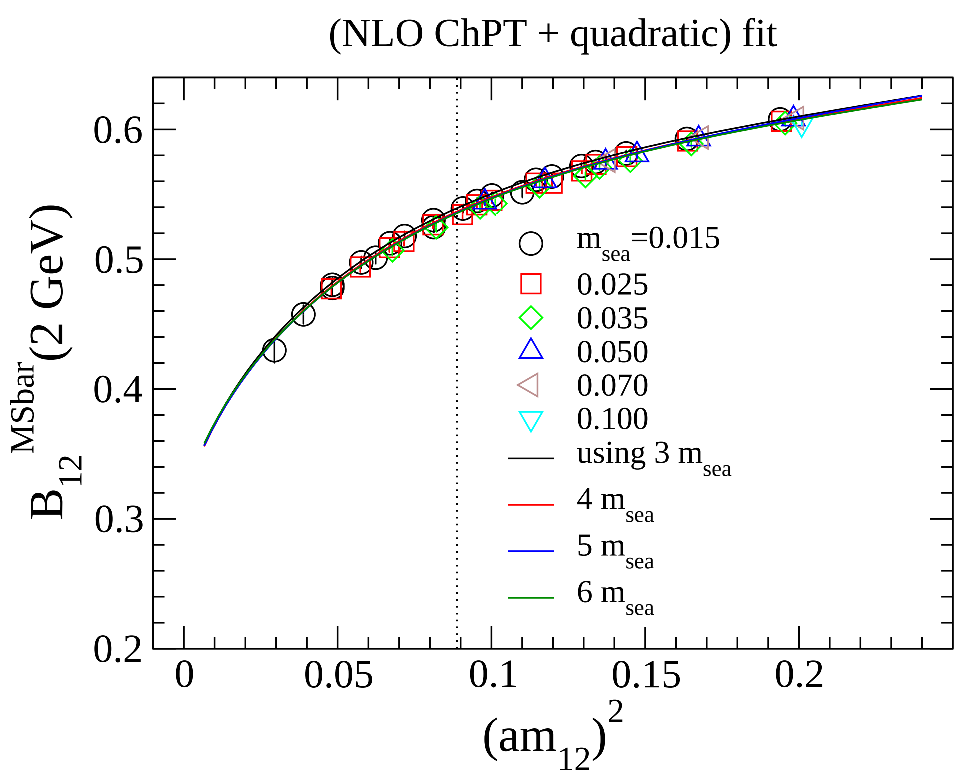
$$B_{\mathrm{K}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.534(52)$$

JLQCD Collaboration (Aoki et al.) Phys Rev D77 (2008) 094503

- $N_\mathrm{f} = 2$, overlap fermions
- Single lattice spacing: $a = 0.12\,\mathrm{fm}$ (from static potential)
- Pion masses: $m_\pi \gtrsim 290\,\mathrm{MeV}$
- Volume: $16^3 \cdot 32, \quad L \simeq 1.9\,\mathrm{fm}, \quad m_\pi^{\mathsf{min}} L = 2.75$
- Non-perturbative renormalisation via RI/MOM scheme
- Perform calculation in sectors of fixed topology

JLQCD Collaboration (Aoki et al.) Phys Rev D77 (2008) 094503

- Chiral fits based on NLO partially quenched ChPT plus analytic term



$$B_K^{\overline{MS}}(2\,\text{GeV}) = 0.537(4)(40), \qquad \widehat{B}_K = 0.758(6)(71)$$

Aubin, Laiho & Van de Water, Phys Rev D81 (2010) 014507

- $N_\mathrm{f} = 2 + 1$, domain wall valence fermions on rooted staggered sea quarks

$\rightarrow$ "mixed action"

- Two lattice spacings: $a \approx 0.09, 0.12\,\mathrm{fm}$ (from static potential)

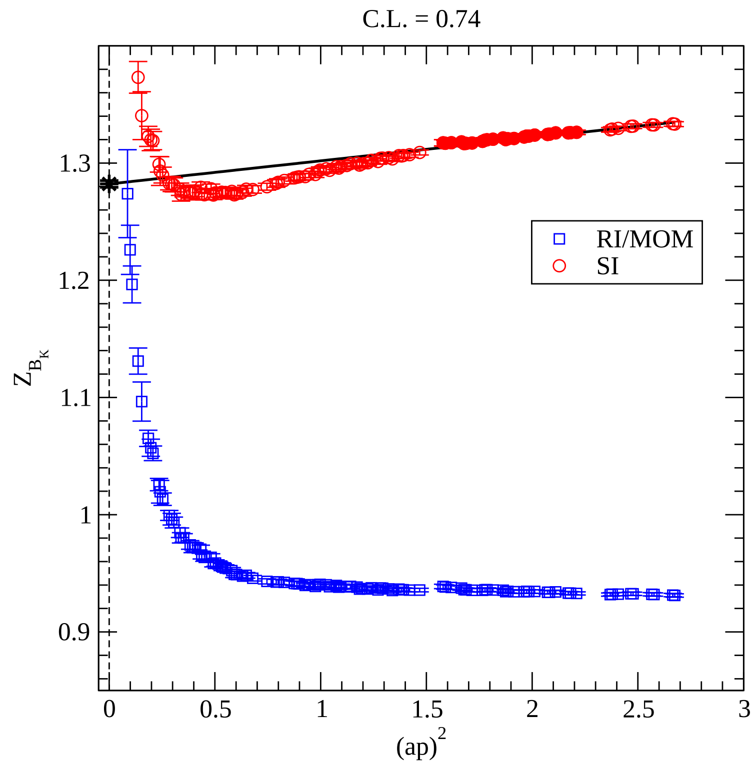
- Pion masses: $m_\pi \gtrsim 230\,\mathrm{MeV}$

- Volume: $m_\pi^{\mathrm{min}} L \gtrsim 4, \quad L \simeq 2.3 - 3.4\,\mathrm{fm}$

- Non-perturbative renormalisation via RI/MOM scheme

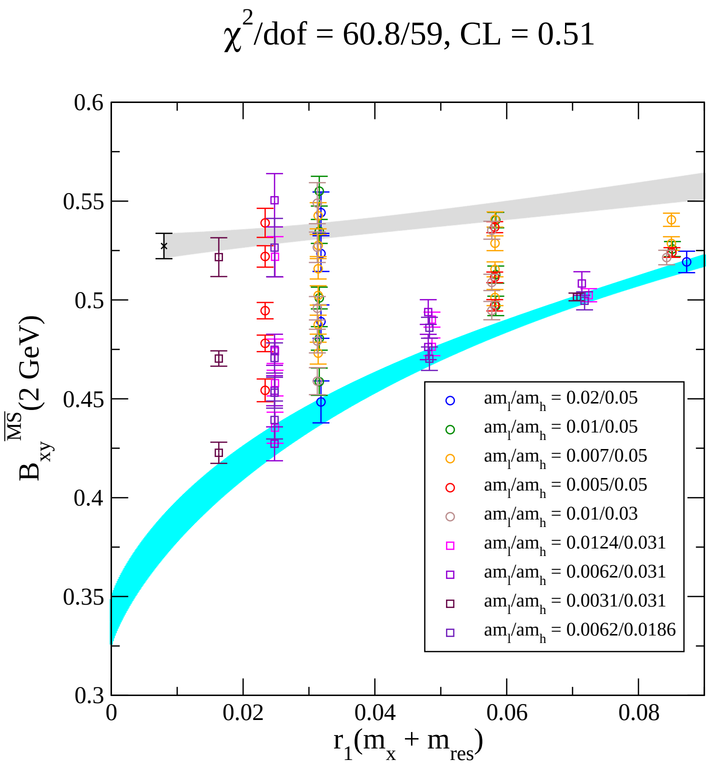
Aubin, Laiho & Van de Water, Phys Rev D81 (2010) 014507

- Largest uncertainty: matching between RI/MOM scheme and RGI



Aubin, Laiho & Van de Water, Phys Rev D81 (2010) 014507

- Chiral fits: extract B_{K} and B_{K}^0



$$B_{\mathrm{K}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 0.527(6)(22), \qquad \widehat{B}_{\mathrm{K}} = 0.724(8)(29)$$

Compilation & Comparison: The FLAG Working Group

- Rate lattice results according to systematic error analysis

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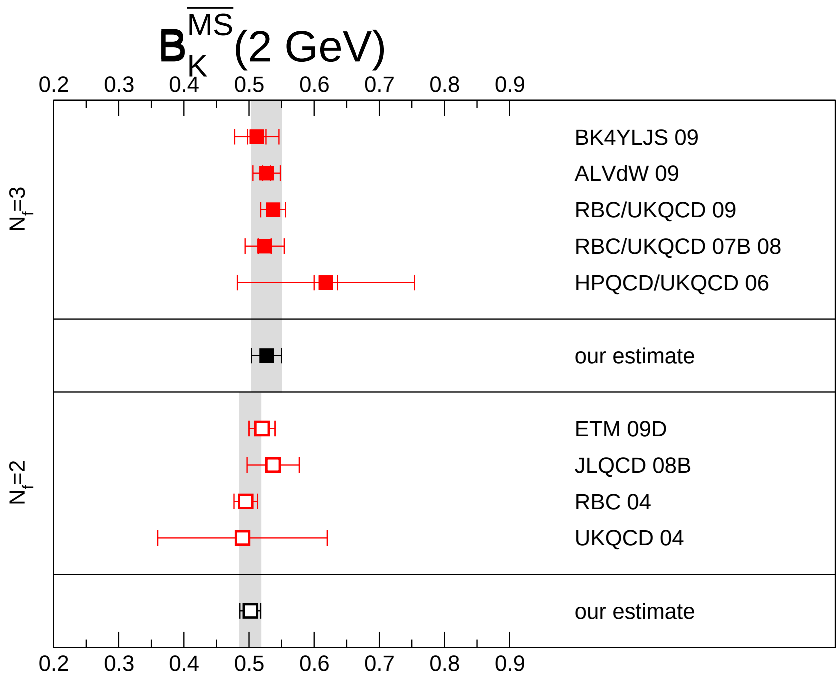
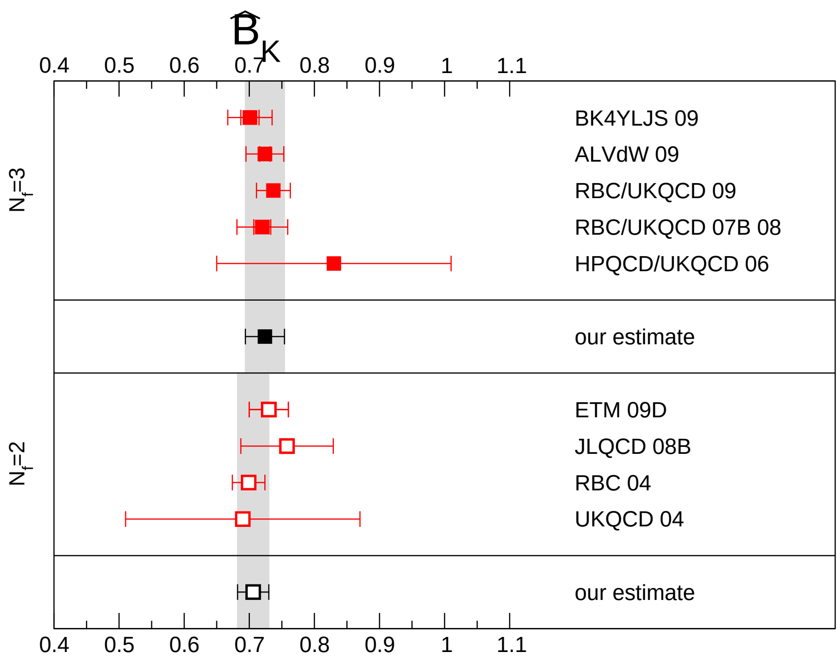
Collaboration	N_f	publication status	continuum extrapolation	chiral extrapolation	finite volume	renormalization	running	$B_\mathrm{K}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})$	$\hat{B}_\mathrm{K}$
BK4YLJS 09	2+1	C	● (green)	● (orange)	● (orange)	● (red)	● (orange)	0.512(14)(34)	0.701(19)(47)
ALVdW 09	2+1	A	● (orange)	● (green)	● (orange)	● (green)	● (orange)	0.527(6)(21)	0.724(8)(29)
RBC/UKQCD 09	2+1	C	● (orange)	● (orange)	● (green)	● (green)	● (orange)	0.537(19)	0.737(26)
RBC/UKQCD 07B, 08	2+1	A	● (red)	● (orange)	● (green)	● (green)	● (orange)	0.524(10)(28)	0.720(13)(37)
HPQCD/UKQCD 06	2+1	A	● (red)	● (orange)	● (green)	● (red)	● (orange)	0.618(18)(135)	0.83(18)
ETM 09D	2	C	● (green)	● (orange)	● (orange)	● (green)	● (orange)	0.52(2)(2)	0.73(3)(3)
JLQCD 08B	2	A	● (red)	● (orange)	● (red)	● (green)	● (orange)	0.537(4)(40)	0.758(6)(71)
RBC 04	2	A	● (red)	● (red)	● (red)	● (green)	● (orange)	0.495(18)	0.699(25)
UKQCD 04	2	A	● (red)	● (red)	● (red)	● (red)	● (orange)	0.49(13)	0.69(18)

Compilation & Comparison: The FLAG Working Group

- Provide "global" averages





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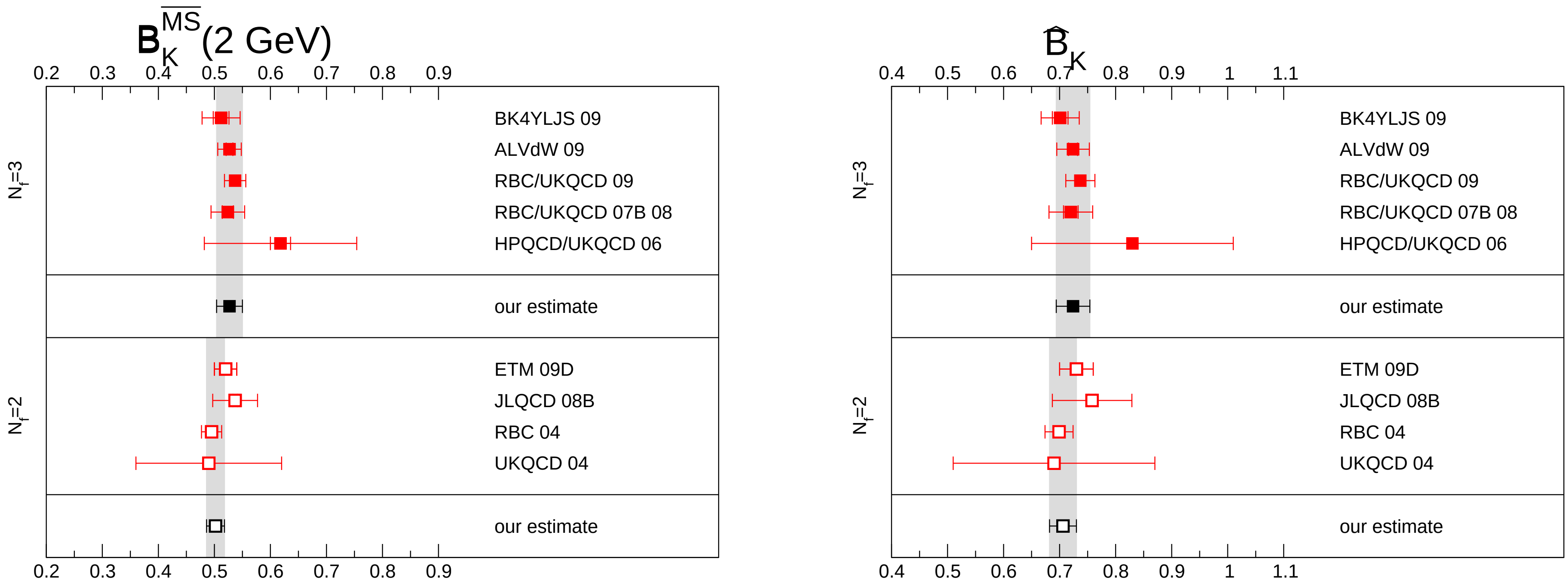


Weighted averages (published results only — preliminary!):

$$B_K^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.502(16), \qquad \widehat{B}_{\rm K} = 0.706(24), \qquad N_{\rm f} = 2$$

$$B_K^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.527(18), \qquad \widehat{B}_{\rm K} = 0.724(24), \qquad N_{\rm f} = 2+1$$

First row unitarity and $|V_{us}|$

- Unitarity of CKM matrix implies

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1, \qquad |V_{ub}| = 3.9(4) \cdot 10^{-3}$$

- $|V_{ud}| = 0.97425 \pm 0.00022$ from super-allowed nuclear $0^+ \to 0^+$ β-decays

- Precision data on branching fractions of kaon decays yield:

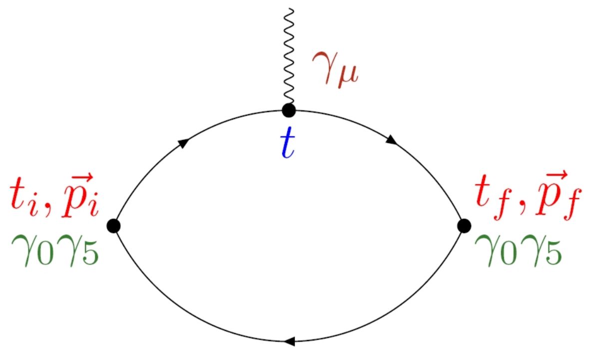
$$|V_{us}| f_+(0) = 0.21661(47), \quad \left|\frac{V_{us} f_{\rm K}}{V_{ud} f_\pi}\right| = 0.27599(59)$$

$\longrightarrow$ Assuming first row unitarity implies

$$|V_{us}| = 0.22544(95), \quad f_+(0) = 0.9608(46), \quad f_{\rm K}/f_\pi = 1.1927(59)$$

- Can we perform a precision test of CKM unitarity using lattice and experimental data alone?

$K_{\ell 3}$ decays on the lattice



$$\sum_{\vec{x}_i, \vec{x}_f} \mathrm{e}^{-i(\vec{x}_f - \vec{x})\cdot\vec{p}_f}\, \mathrm{e}^{-i(\vec{x}-\vec{x}_i)\cdot\vec{p}_i} \left\langle \phi_\pi(x_f) V_\mu(x) \phi_{\mathrm{K}}^\dagger(x_i) \right\rangle$$

$$\langle \pi(\vec{p}_\pi) | V_\mu(0) | \mathrm{K}(\vec{p}_K) \rangle = f_+(q^2)(p_K + p_\pi)_\mu + f_-(q^2)(p_K - p_\pi)_\mu$$

- Lattice momenta: $\vec{p}_i,\, \vec{p}_f = (0,0,0),\, (1,0,0), \ldots, (1,1,0), \ldots$ times $\dfrac{2\pi}{L}$
- Must interpolate lattice results to $q^2 = 0$ $\Rightarrow$ introduce model dependence
- Aim for better momentum resolution

Twisted boundary conditions

[Bedaque 2004; de Divitiis, Petronzio & Tantalo 2004; Flynn, Jüttner & Sachrajda 2005]

- Apply "twisted" spatial boundary conditions;

  Impose periodicity up to a phase $\vec{\theta}$:

$$\psi(x + L\hat{e}_k) = \mathrm{e}^{i\theta_k}\psi(x) \quad \Rightarrow \quad p_k = n_k\frac{2\pi}{L} + \frac{\theta_k}{L}, \quad k = 1, 2, 3$$

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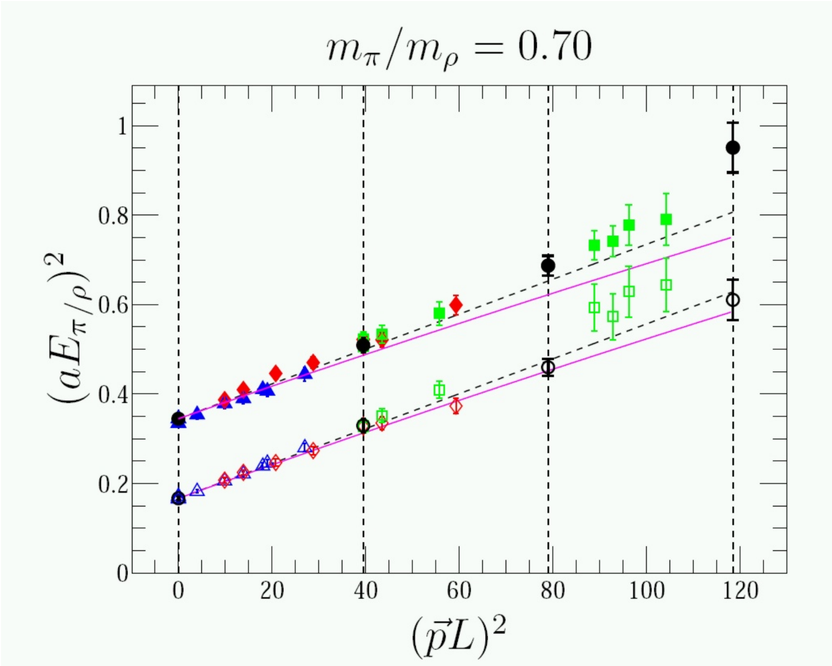
- Check dispersion relation:



[Flynn, Jüttner, Sachrajda, hep-lat/0506016]

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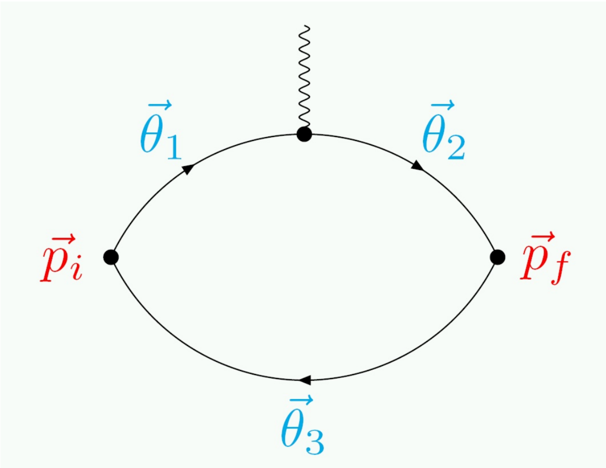
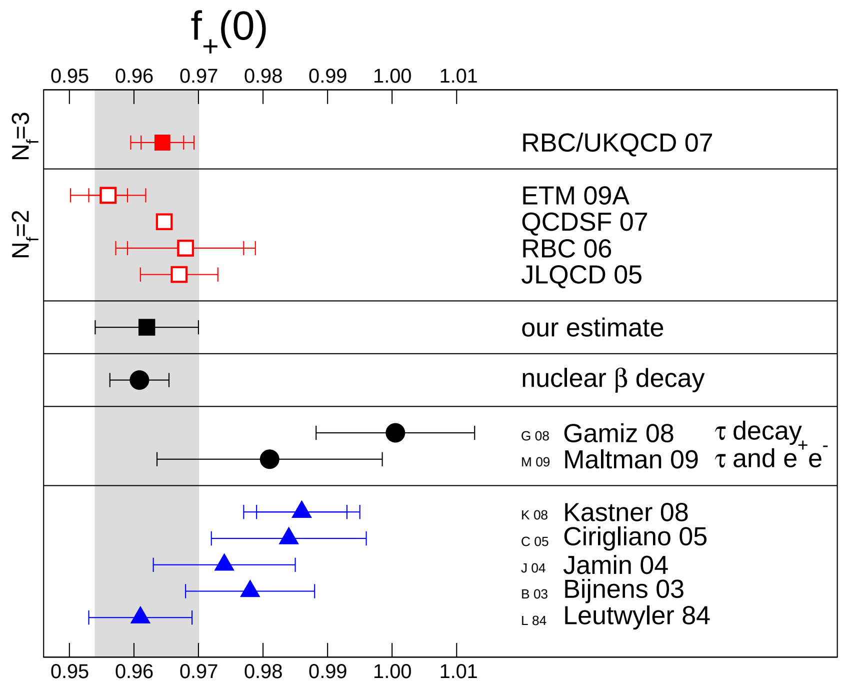
- Can tune $|q^2|$ to any desired value:



$$\vec{\theta}_i = \vec{\theta}_1 - \vec{\theta}_3,$$
$$\vec{\theta}_f = \vec{\theta}_2 - \vec{\theta}_3$$

$$\Rightarrow \; q^2 = (p_K - p_\pi)^2 = \Big(E_K(\vec{p}_i) - E_\pi(\vec{p}_f)\Big)^2 - \left[\left(\vec{p}_i + \frac{\vec{\theta}_i}{L}\right) - \left(\vec{p}_f + \frac{\vec{\theta}_f}{L}\right)\right]^2$$

Results for $f_+(0)$ (FLAG Working Group)



- Lattice data consistent with determination via Ademollo-Gatto theorem *[Leutwyler + Roos 1984]*

$$f_+(0) = 0.964(3)(4) \quad \Rightarrow \quad |V_{us}| = 0.2247(13), \quad N_\mathrm{f} = 2+1$$

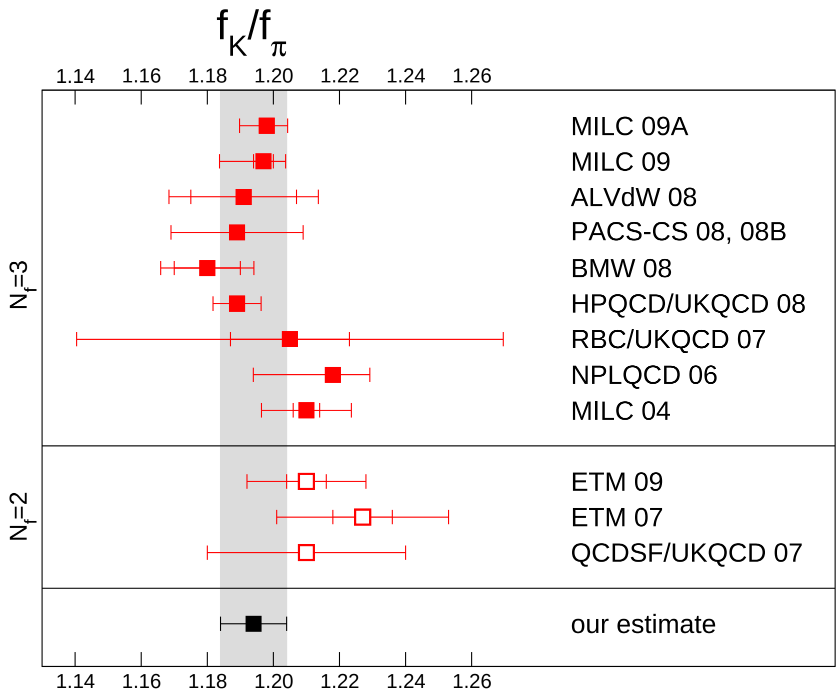
$$f_+(0) = 0.956(6)(6), \quad N_\mathrm{f} = 2$$

$|V_{us}|$ from $K_{\ell 2}$ decays

[Marciano, Phys Rev Lett 93 (2004) 231803]

- Leptonic decay rate: $$\frac{\Gamma(K\to\mu\bar{\nu}_\mu(\gamma))}{\Gamma(\pi\to e\bar{\nu}_e(\gamma))} \propto \frac{|V_{us}|^2}{|V_{ud}|^2}\frac{f_{\rm K}^2 m_{\rm K}}{f_\pi^2 m_\pi}$$

$\longrightarrow$ Determine the ratio $f_{\rm K}/f_\pi$



- Global averages (FLAG Working Group): (preliminary!)

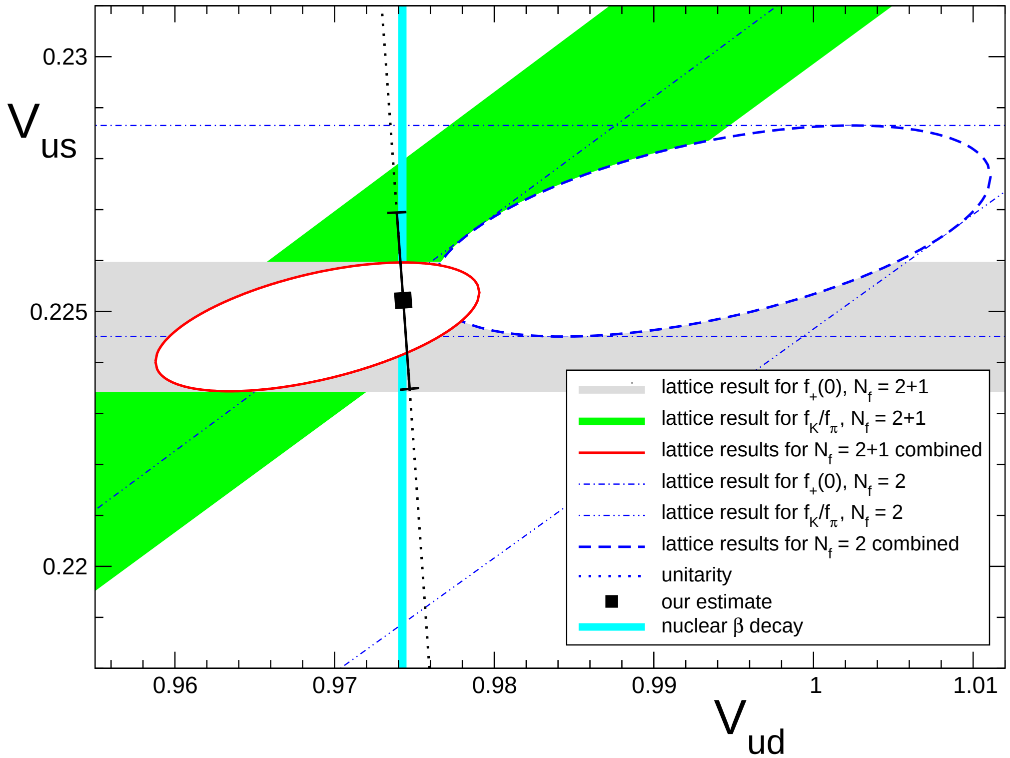
$$f_{\rm K}/f_\pi = 1.190(2)(10), \quad N_{\rm f}=2+1$$

$$f_{\rm K}/f_\pi = 1.210(6)(17), \quad N_{\rm f}=2$$

Test of the Standard Model

[FLAG Working Group — preliminary]

- Lattice results for $K_{\ell 3}$ and $K_{\ell 2}$ decays yield:



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- Lattice results for $K_{\ell 3}$ and $K_{\ell 2}$ decays yield $(N_\mathrm{f} = 2+1)$:

$$|V_{us}| = 0.2247(13), \quad |V_{us}/V_{ud}| = 0.2319(20)$$

$$\Rightarrow \quad |V_{ud}|^2 + |V_{us}|^2 = 0.989(20)$$

(involves only lattice data and measured branching fractions)

- Combining lattice results with $|V_{ud}|$ from nuclear β-decay:

$$f_+(0)\text{-input}: \quad |V_{ud}|^2 + |V_{us}|^2 = 0.9997(7)$$

$$f_\mathrm{K}/f_\pi\text{-input}: \quad |V_{ud}|^2 + |V_{us}|^2 = 1.0002(10)$$

6. Hadronic vacuum polarisation and $(g-2)_\mu$

- Muon anomalous magnetic moment: $a_\mu = \frac{1}{2}(g-2)_\mu$

$$a_\mu = \begin{cases} 11\,659\,208(6.3)\cdot 10^{-10} & \text{Experiment} \\ 11\,659\,179(6.5)\cdot 10^{-10} & \text{SM prediction,} \end{cases} \qquad (3.2\sigma \text{ tension})$$

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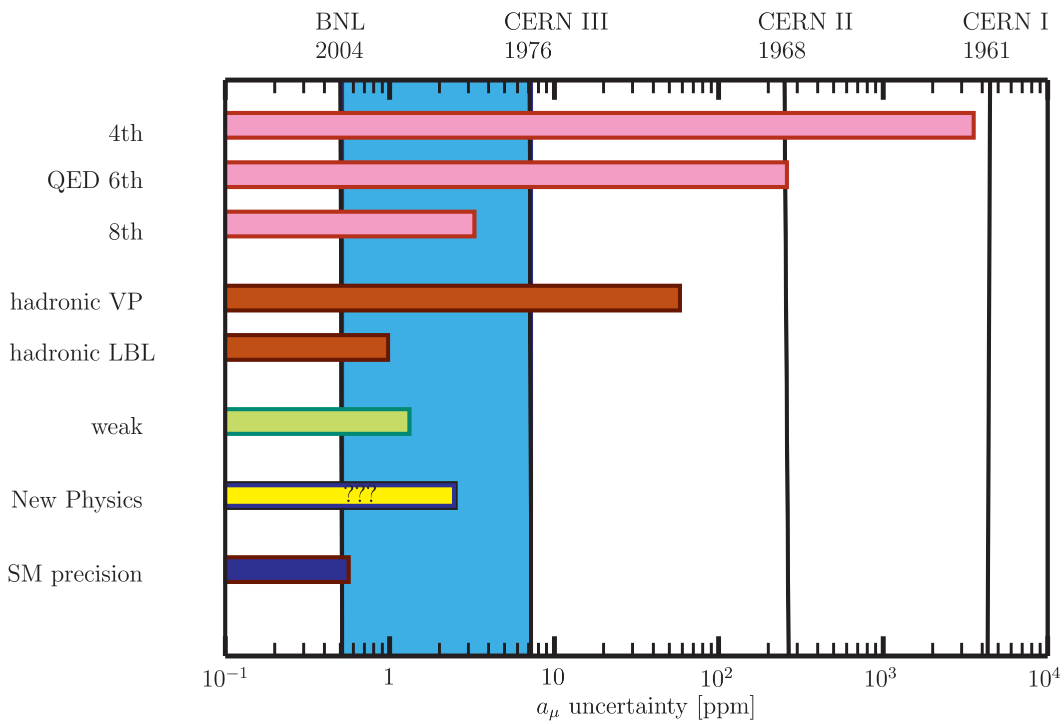
- Experimental sensitivity versus individual contributions:



[Jegerlehner & Nyffeler, arXiv:0902.3360]

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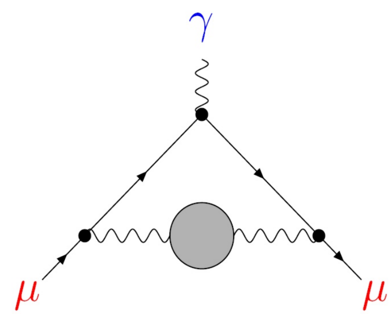
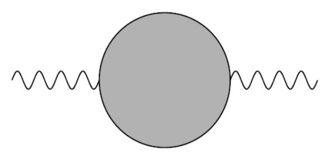
- Hadronic vacuum polarisation; leading contribution:



- Vacuum polarisation tensor:



$$\Pi_{\mu\nu}(q^2) = \int \mathrm{d}^4x\, \mathrm{e}^{iq\cdot(x-y)}\, \langle J_\mu(x) J_\nu(y)\rangle \equiv (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi(q^2)$$

- a_μ^{had} determined from convolution integral:

$$a_\mu^{\mathsf{had}} = 4\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\{\Pi(Q^2) - \Pi(0)\}$$

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Problems for lattice calculations:

- Convolution integral dominated by momenta near m_μ :

  maximum of $f(Q^2)$ located at: $(\sqrt{5}-2)m_\mu^2 \approx 0.003\,\mathrm{GeV}^2$

  lowest momentum transfer: $\left(\frac{2\pi}{T}\right)^2 \approx 0.06\,\mathrm{GeV}^2$

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  – Large noise-to-signal ratio

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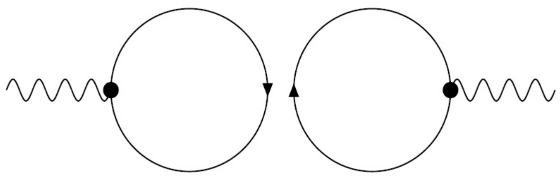

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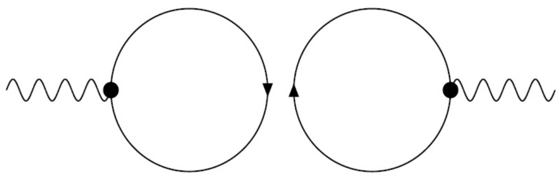

- Resonance effects: $\rho \to \pi\pi$

New strategy for two-flavour QCD

[Della Morte & Jüttner, arXiv:0910.3755]

- QCD with $N_\mathrm{f} = 2$ flavours: $J_\mu(x) = \left(\frac{2}{3} j_\mu^{uu} - \frac{1}{3} j_\mu^{dd}\right)(x)$

$$\langle J_\mu(x) J_\nu(y) \rangle = \frac{4}{9}\left\langle j_\mu^{uu} j_\nu^{uu} \right\rangle - \frac{2}{9}\left\langle j_\mu^{uu} j_\nu^{dd} \right\rangle - \frac{2}{9}\left\langle j_\mu^{dd} j_\nu^{uu} \right\rangle + \frac{1}{9}\left\langle j_\mu^{dd} j_\nu^{dd} \right\rangle$$

- Impose isospin symmetry, $m_u = m_d$, set $y \equiv 0$; Correlation function:

$$C_{\mu\nu}(q) = \frac{5}{9} C_{\mu\nu}^{(\mathrm{con})}(q) + \frac{1}{9} C_{\mu\nu}^{(\mathrm{disc})}(q)$$

$$C_{\mu\nu}^{(\mathrm{con})}(q) = \sum_x \mathrm{e}^{iq\cdot x} \left\langle \mathrm{Tr}\left[\overline{\psi}(x)\gamma_\mu\psi(x)\,\overline{\psi}(0)\gamma_\mu\psi(0)\right] \right\rangle$$

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- $C_{\mu\nu}^{(\mathrm{con})}(q)$ and $C_{\mu\nu}^{(\mathrm{disc})}(q)$ have individual continuum and finite volume limits

- $C_{\mu\nu}^{(\mathrm{con})}(q)$ can be evaluated using twisted boundary conditions

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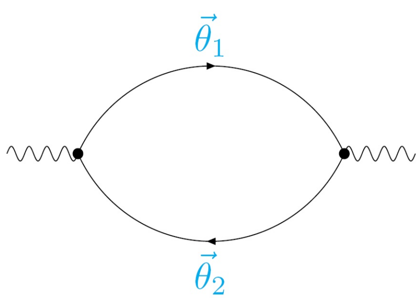

Relative size of the disconnected contribution

[Della Morte & Jüttner, arXiv:0910.3755]

- Compute polarisation tensor in SU(2) ChPT @ NLO
- Determine disconnected and connected contributions to $\Pi(q^2) - \Pi(0)$ (enters convolution integral)

$$\Rightarrow \quad \frac{\Pi^{(\mathrm{disc})}(q^2) - \Pi^{(\mathrm{disc})}(0)}{\Pi^{(\mathrm{con})}(q^2) - \Pi^{(\mathrm{con})}(0)} = -\frac{1}{10}$$

$\longrightarrow$ Effect of disconnected contribution estimated to be a 10% downward shift

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Strategy to compute a_μ^{had} in two-flavour QCD

- Compute connected contribution using twisted boundary conditions
- Compute disconnected contribution for Fourier modes only:

  $\longrightarrow$ validate its relative suppression predicted by ChPT

Preliminary results

[Della Morte, Jäger, Jüttner, H.W.]

- Test runs at $\beta = 5.3$, $a = 0.069(2)\,\mathrm{fm}$, $32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

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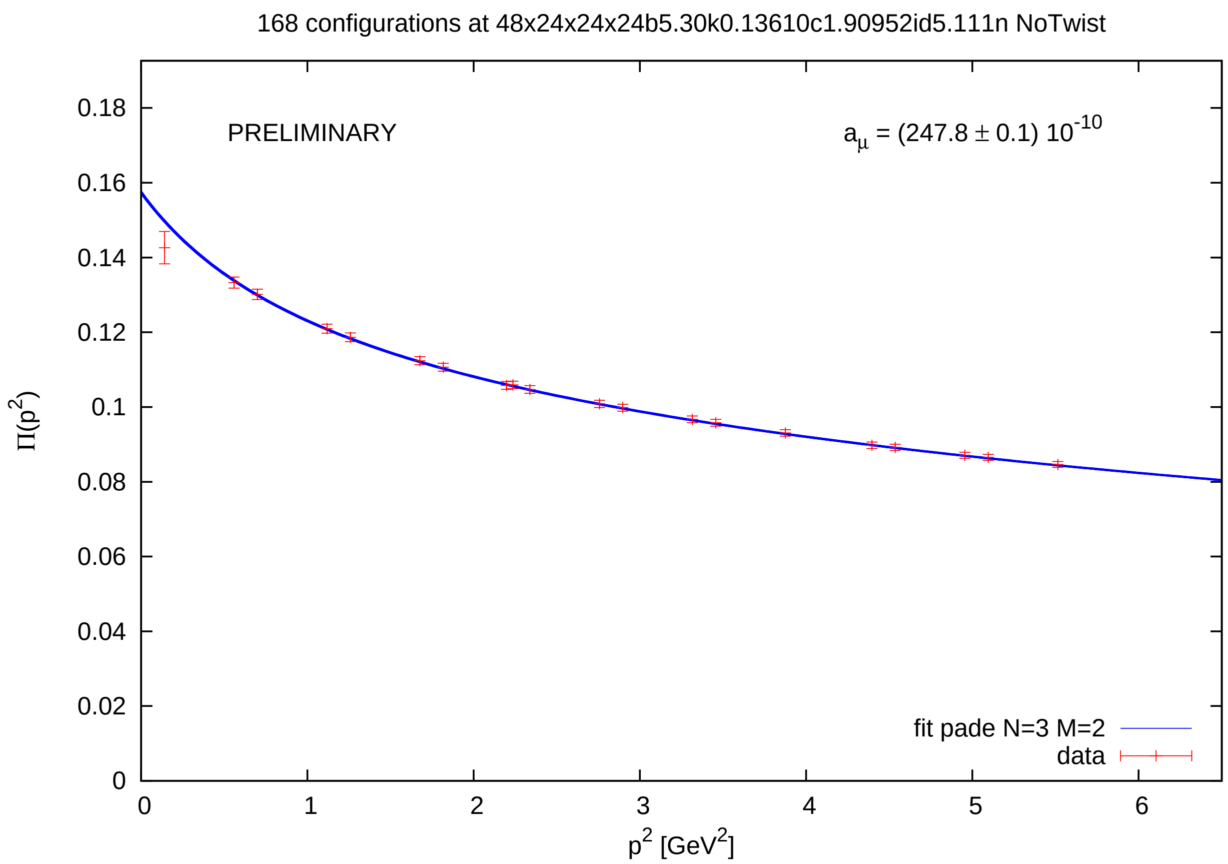
  $m_\pi = 550\,\mathrm{MeV}$, $L \simeq 1.7\,\mathrm{fm}$



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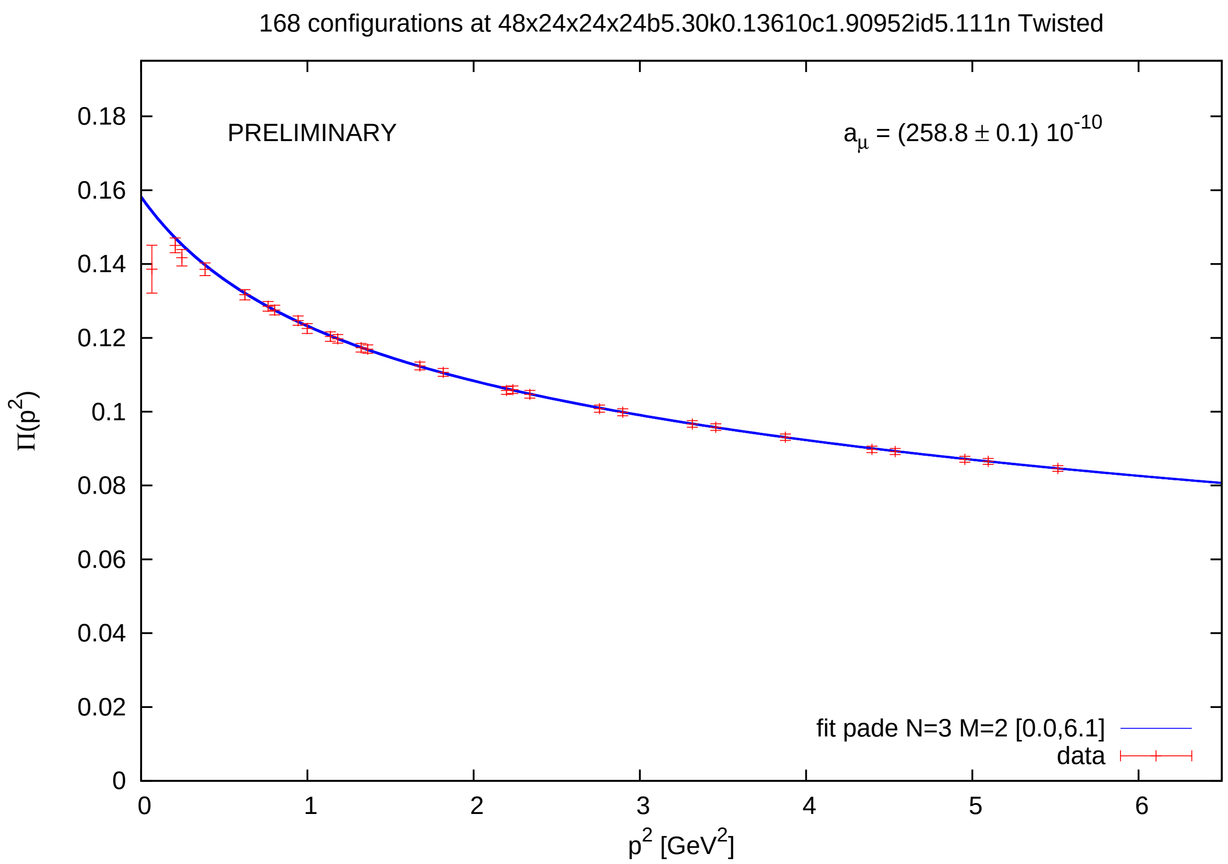

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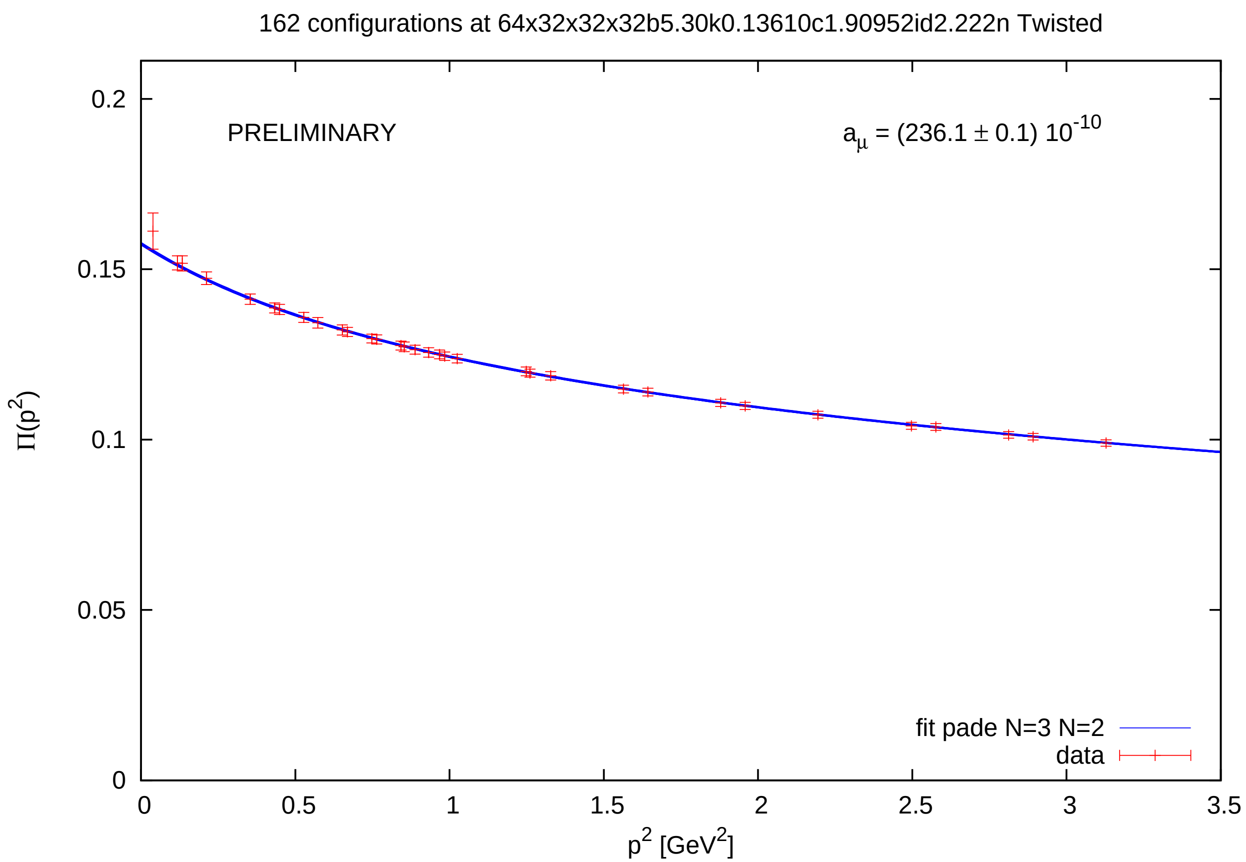

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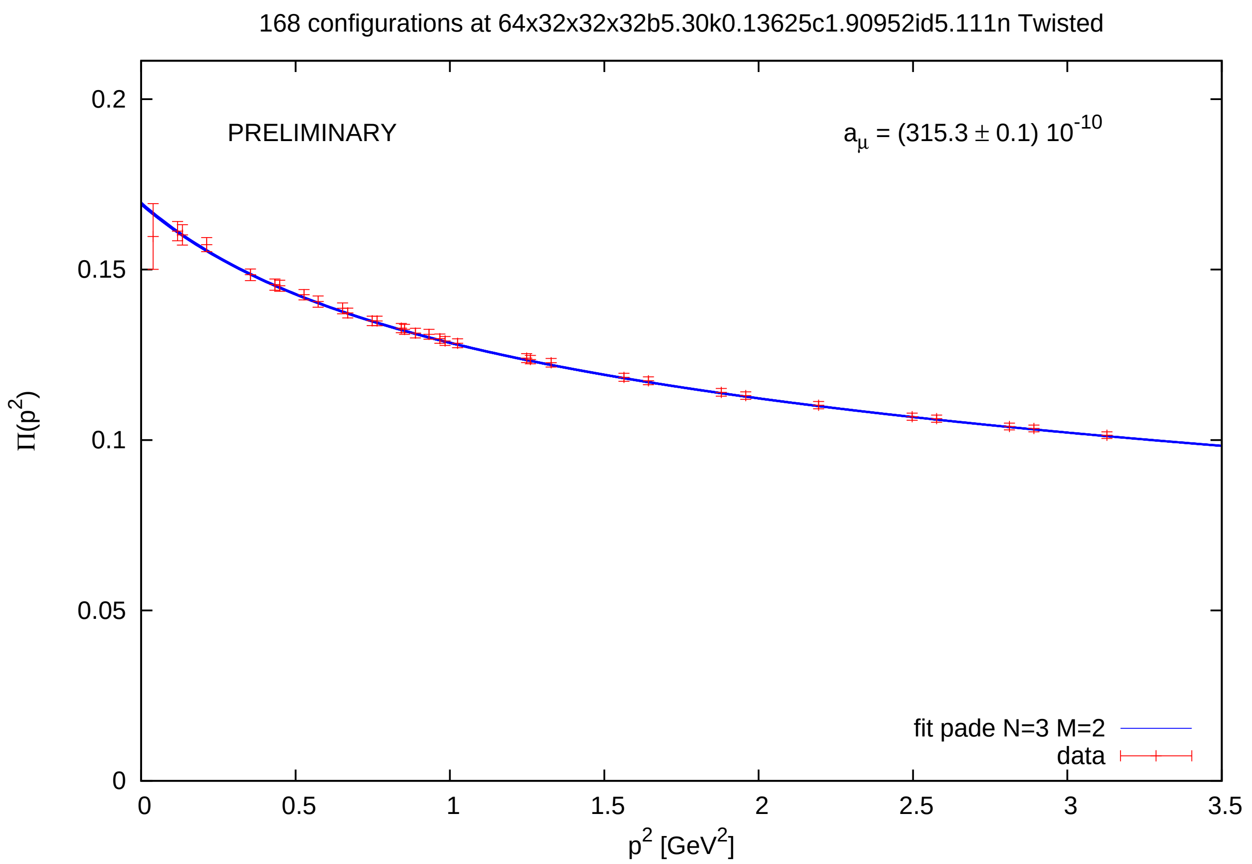
- Test runs at $\beta = 5.3, \quad a = 0.069(2)\,\mathrm{fm}, \quad 32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

  $m_\pi = 420\,\mathrm{MeV}, \quad L \simeq 2.2\,\mathrm{fm}$



Preliminary results

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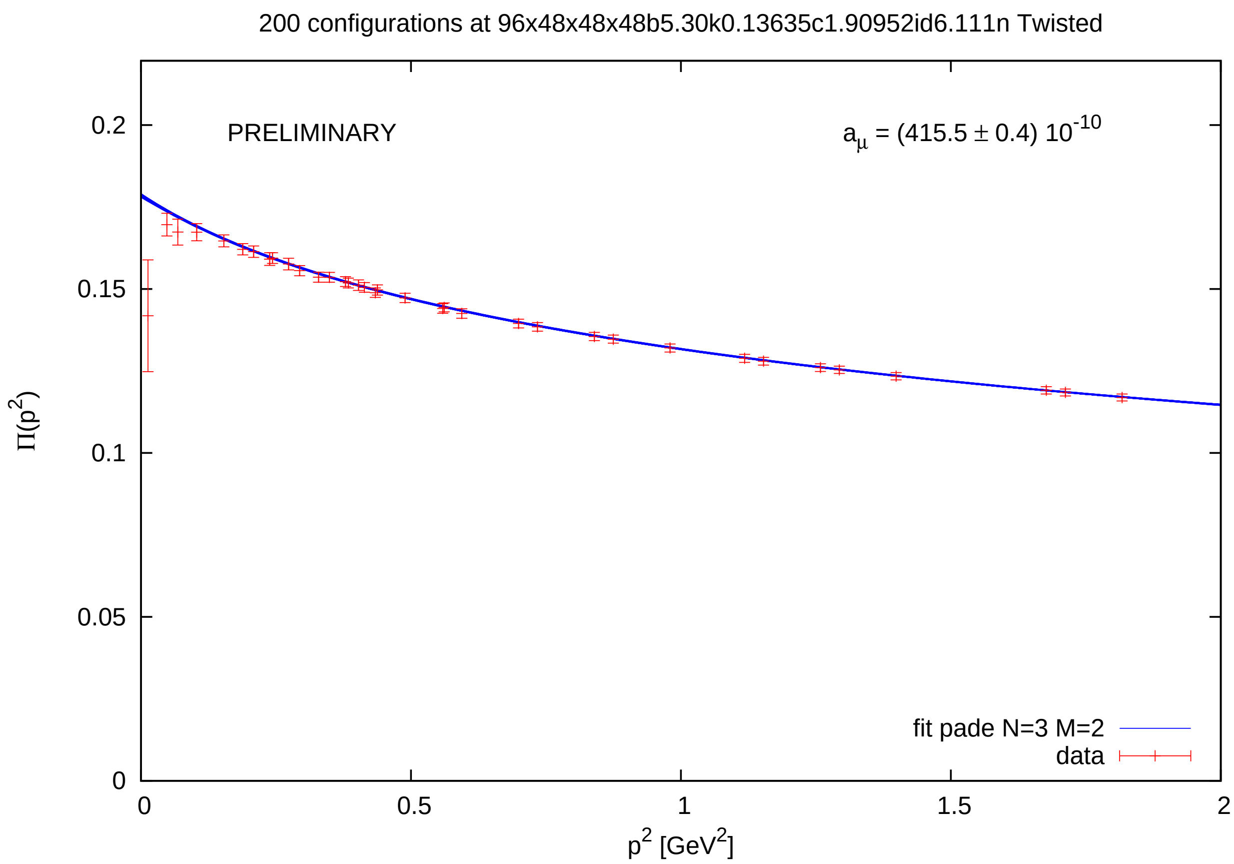
- Test runs at $\beta = 5.3, \quad a = 0.069(2)\,\mathrm{fm}, \quad 32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

$$m_\pi = 290\,\mathrm{MeV}, \quad L \simeq 3.4\,\mathrm{fm}$$



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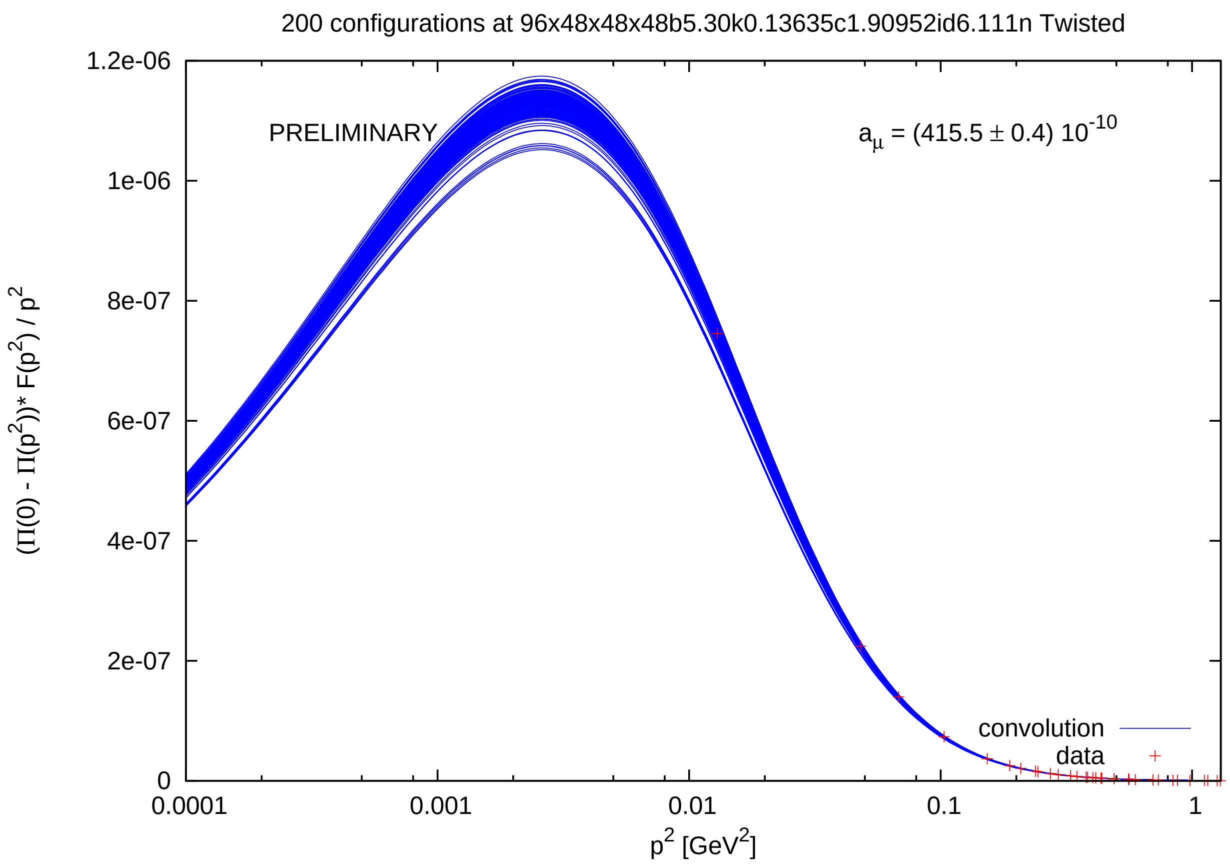
- Test runs at $\beta = 5.3, \quad a = 0.069(2)\,\mathrm{fm}, \quad 32{\times}24^3$, $64{\times}32^3$ and $96{\times}48^3$

  $m_\pi = 290\,\mathrm{MeV}, \quad L \simeq 3.4\,\mathrm{fm}$



Preliminary results

[Della Morte, Jäger, Jüttner, H.W.]

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- Strong pion mass dependence; noticeable finite-volume effects

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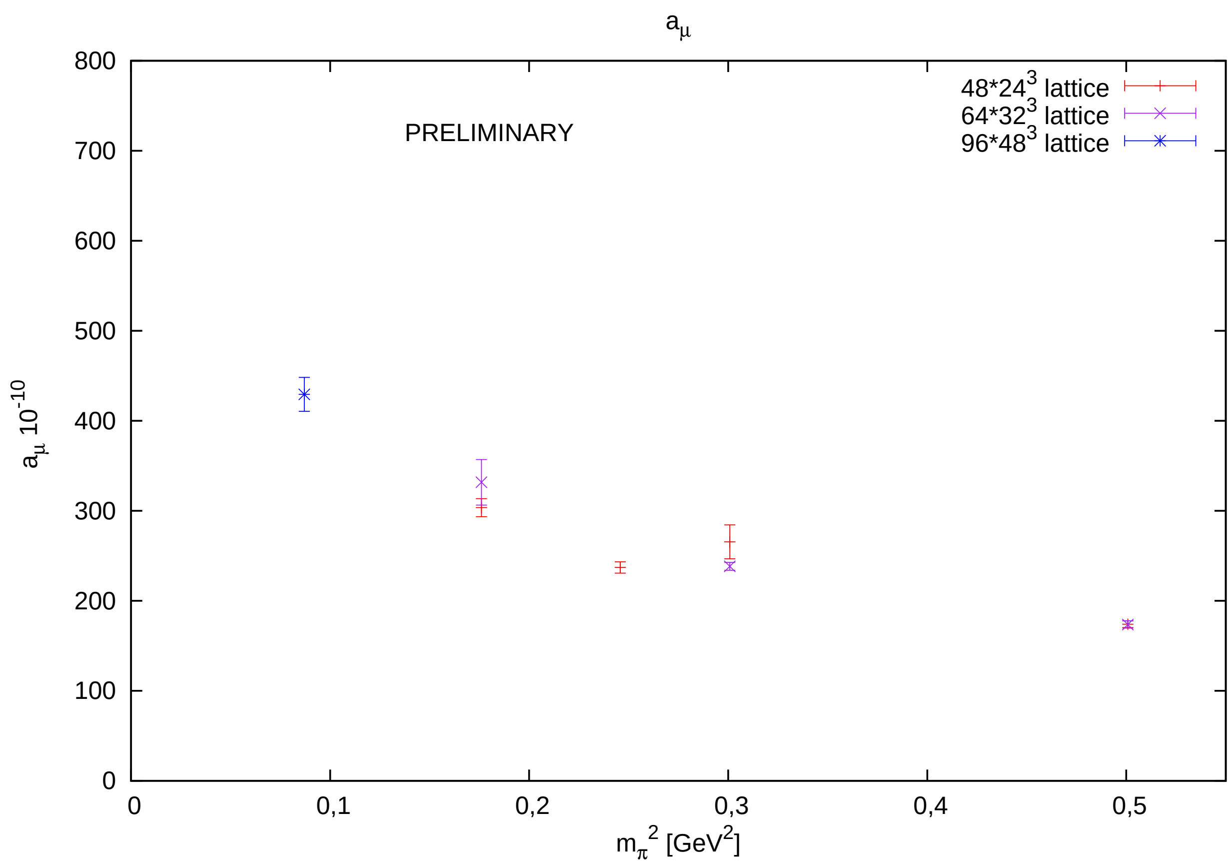

Summary

- Simulations of lattice QCD yield quantitative information with controlled systematic uncertainties:
  - Ground state mesons and baryons
  - Meson decay constants, form factors and mixing parameters
  - Quark masses

- Lattice results are beginning to challenge the conventional phenomenological approach:
  - Model-independent determination of CKM elements
  - Tests of SM unitarity

- Improved control over systematic effects necessary for other quantities, e.g.
  - Hadronic vacuum polarisation contribution to $(g-2)_\mu$
  - Nucleon form factors and structure functions
  - ...