Effective Field Theories

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Lecture III: NR EFT: Apps

- NR effective Lagrangians
- More scattering theory
- Power counting in two-body EFT
- Bound states in EFT

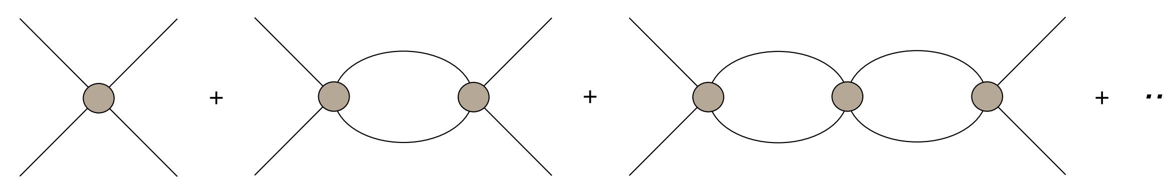
Interacting fermions (nucleons or atoms)

☆ Consider NR scattering theory in d=4

$$S = e^{i\delta(p)} = 1 + i\frac{Mp}{2\pi}\mathcal{A}_2(p)$$

$$\mathcal{A}_2(p) = \frac{4\pi}{M}\frac{1}{p\cot\delta(p) - ip}$$

=



Assume finite range interactions: *Effective Range Theory*

$$p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}\Lambda^2\sum_{n=0}^{\infty} r_n\left(\frac{p^2}{\Lambda^2}\right)^{n+1} = -\frac{1}{a} + \frac{1}{2}r_0 p^2 + \ldots$$

Generally two scenarios:

"Natural" $|a| \sim \Lambda^{-1} \quad , \quad |r_n| \sim \Lambda^{-1}$

$$\mathcal{A}_2 = -\frac{4\pi a}{M}[1 - iap + (ar_0/2 - a^2)p^2 + O(p^3/\Lambda^3)]$$

"Unnatural" $|a| \gg \Lambda^{-1} \quad , \quad |r_n| \sim \Lambda^{-1}$

$$\mathcal{A}_2 = -\frac{4\pi}{M}\frac{1}{(1/a + ip)}[1 + \frac{r_0/2}{(1/a + ip)}p^2 + \frac{(r_0/2)^2}{(1/a + ip)^2}p^4 + \frac{(r_1/2\Lambda^2)}{(1/a + ip)}p^4 + \ldots]$$

Now we will reproduce these two scenarios using EFT:

(I) Identify low-energy d.o.f

(II) Identify the symmetries

(III) Construct most general EFT

(IV) Determine *power counting*

(V) Determine parameters (*matching to EXP*)

Interacting Fermions (nucleons or atoms)

Assume: finite range interaction in four space-time dimensions

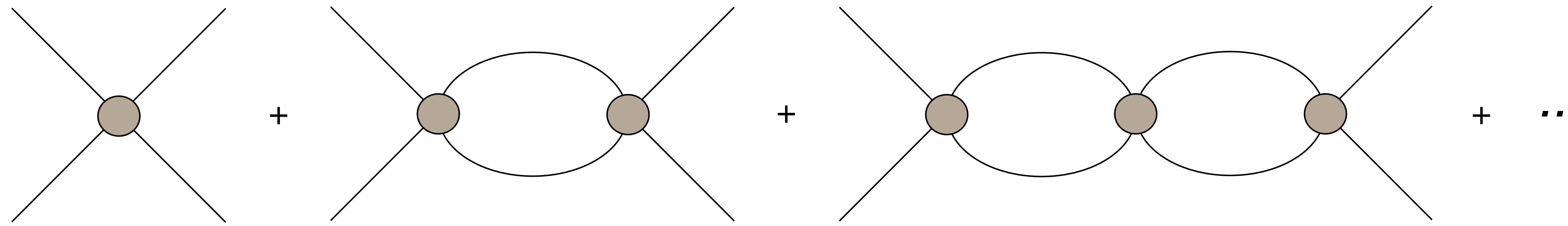
$$\mathcal{L}_{EFT} = N^\dagger \left(i\partial_t + \frac{\nabla^2}{2M} \right) N + C_0 (N^\dagger N)^2 + \frac{C_2}{8} \left[(NN)^\dagger (N \overleftrightarrow{\nabla}^2 N) + h.c \right] + \ldots$$

<u>Note</u>: can choose alternate basis:

$$C_0 \left(N^T \mathcal{P}_x N \right)^\dagger \left(N^T \mathcal{P}_x N \right)$$

Projection operator onto given channel: $\mathcal{P}_x$

As before, can solve exactly (formally)

$\mathcal{A}_2 =$ 

$$\mathcal{A}_2(p) = -\frac{\sum C_{2n}\, p^{2n}}{1 - I_0(p) \sum C_{2n}\, p^{2n}}$$

Dimensional regularization: $\epsilon \equiv 4 - D$

$$\begin{aligned}
I_n \equiv &\quad i(\mu/2)^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\mathbf{q}^{2n}}{\left(E/2 + q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon\right)\left(E/2 - q_0 - \frac{\mathbf{q}^2}{2M} + i\epsilon\right)} \\
= &\quad M(\mu/2)^\epsilon \int \frac{\mathrm{d}^{(D-1)}\mathbf{q}}{(2\pi)^{(D-1)}}\, \mathbf{q}^{2n} \left(\frac{1}{p^2 - \mathbf{q}^2 + i\epsilon}\right) \\
= &\quad -Mp^{2n}(-p^2 - i\epsilon)^{(D-3)/2}\Gamma\left(\frac{3-D}{2}\right) \frac{(\mu/2)^\epsilon}{(4\pi)^{(D-1)/2}}
\end{aligned}$$

"Natural" $|a| \sim \Lambda^{-1} \quad , \quad |r_n| \sim \Lambda^{-1}$

$$I_n^{\overline{MS}} = -i\left(\frac{M}{4\pi}\right)p^{2n+1}$$

$$\mathcal{A}_2 = -\frac{\sum C_{2n}p^{2n}}{1 + i(Mp/4\pi)\sum C_{2n}p^{2n}} = \sum_{n=0}^{\infty}(\mathcal{A}_2)_n \qquad (\mathcal{A}_2)_n \sim \mathcal{O}(p^n)$$

$$(\mathcal{A}_2)_0 = -C_0 \ , \qquad (\mathcal{A}_2)_1 = iC_0^2\frac{Mp}{4\pi} \ , \qquad (\mathcal{A}_2)_2 = C_0^3\left(\frac{Mp}{4\pi}\right)^2 - C_2p^2$$

This amplitude must match to ERT amplitude!

Match to:

$$\mathcal{A}_2 = -\frac{4\pi a}{M}[1 - iap + (ar_0/2 - a^2)p^2 + O(p^3/\Lambda^3)]$$

$$C_0 = \frac{4\pi a}{M}$$

$$C_2 = C_0 \frac{ar_0}{2}$$

In general:

$$C_{2n} \sim \frac{4\pi}{M\Lambda}\frac{1}{\Lambda^{2n}}$$

Power counting

- propagator $1/p^2$
- loop integration $\int d^4q \;\rightarrow\; p^5$
- vertex $C_{2n}\nabla^{2n} \;\rightarrow\; p^{2n}$

“Unnatural” $|a| \gg \Lambda^{-1} \quad , \quad |r_n| \sim \Lambda^{-1}$

Realistic case for nuclear physics!

Experiment:

$$
\begin{aligned}
a_s^{^1S_0} &= -23.714 \text{ fm} \qquad & r_s^{^1S_0} &= 2.73 \text{ fm} \\
a_s^{^3S_1} &= 5.425 \text{ fm} \qquad & r_s^{^3S_1} &= 1.749 \text{ fm}
\end{aligned}
$$

$$a_s \gg \Lambda^{-1} \sim m_\pi^{-1}$$

Now we want:

$$\mathcal{A}_2 = -\frac{4\pi}{M}\frac{1}{(1/a+ip)}[1+\frac{r_0/2}{(1/a+ip)}p^2+\frac{(r_0/2)^2}{(1/a+ip)^2}p^4+\frac{(r_1/2\Lambda^2)}{(1/a+ip)}p^4+\ldots]$$

$$\sum_{n=-1}^{\infty} (\mathcal{A}_2)_n \qquad (\mathcal{A}_2)_n \sim \mathcal{O}(p^n)$$

$$(\mathcal{A}_2)_{-1} \;=\; -\frac{4\pi}{M}\frac{1}{(1/a+ip)} \xrightarrow{C_0 \;=\; \frac{4\pi a}{M}} \frac{-C_0}{1+\frac{C_0 M}{4\pi}ip}$$

Need new *power counting*! $\qquad C_0 \;\sim\; \frac{1}{p}\;!$

Clever trick: subtract pole in D=3 dimensions!

$$\delta I_n \;=\; -\frac{M p^{2n}\mu}{4\pi(D-3)}$$

$$I_n^{PDS} = I_n + \delta I_n = -p^{2n}\left(\frac{M}{4\pi}\right)(\mu + ip)$$

Match to:

$$\mathcal{A}_2 = -\frac{4\pi}{M}\frac{1}{(1/a+ip)}[1 + \frac{r_0/2}{(1/a+ip)}p^2 + \frac{(r_0/2)^2}{(1/a+ip)^2}p^4 + \frac{(r_1/2\Lambda^2)}{(1/a+ip)}p^4 + \ldots]$$

$$C_0(\mu) = \frac{4\pi}{M}\left(\frac{1}{-\mu+1/a}\right)$$

$$C_2(\mu) = \frac{4\pi}{M}\left(\frac{1}{-\mu+1/a}\right)^2\frac{r_0}{2}$$

In general:

$$C_{2n}(\mu) \sim \frac{4\pi}{M\Lambda^n\mu^{n+1}}$$

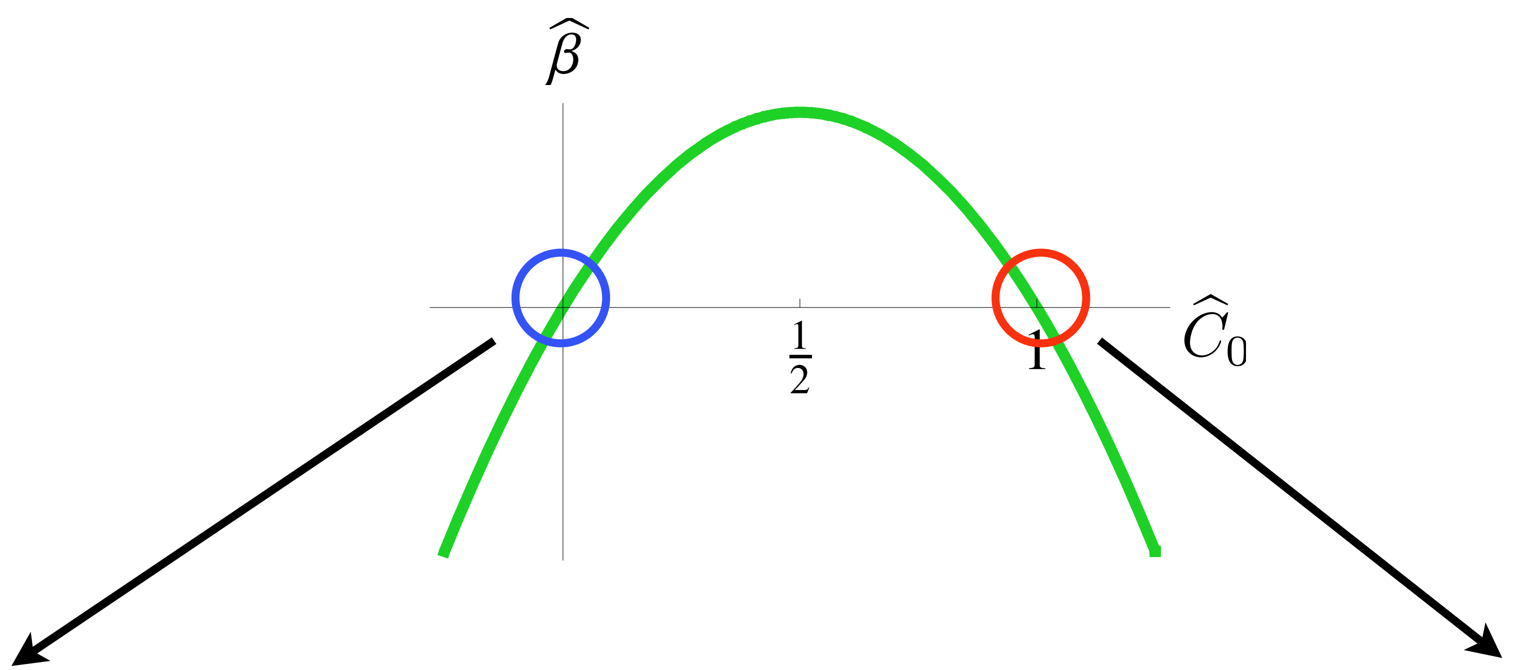
Power counting

- propagator $1/p^2$
- loop integration $\int d^4q \;\rightarrow\; p^5$
- vertex $C_{2n}\nabla^{2n} \;\rightarrow\; p^{n-1}$
- rg scale $\mu \sim p$

Renormalization group interpretation

Define: $$\hat{C}_0 \equiv -\frac{M\mu}{4\pi} C_0 \;=\; \frac{\mu}{\mu - 1/a}$$

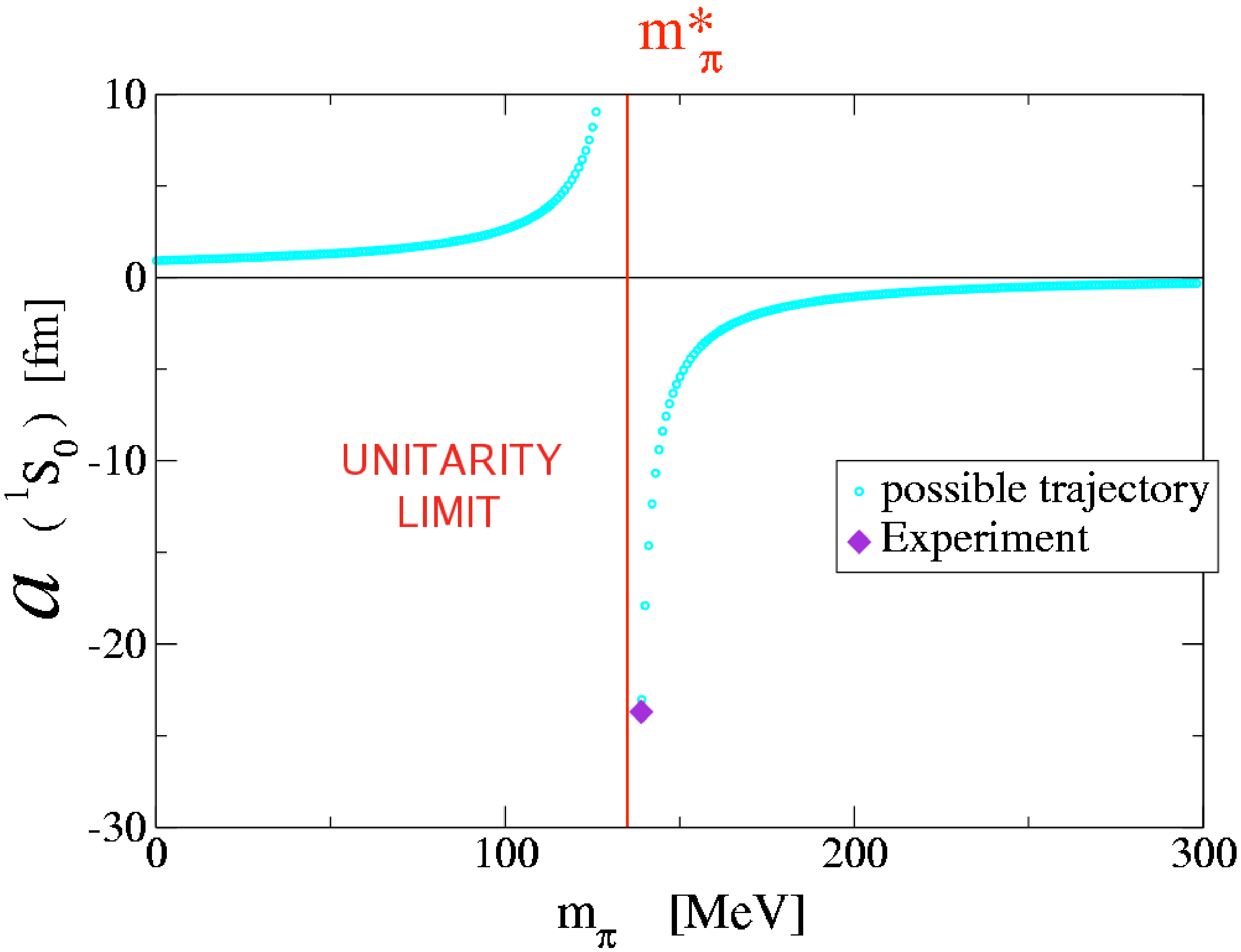
$$\hat{\beta}_0 \;=\; \mu \frac{d}{d\mu} \hat{C}_0 \;=\; -\hat{C}_0(\hat{C}_0 - 1)$$



Trivial IR fixed point:
"natural case"

Nontrivial UV fixed point:
"unnatural case"

Why is nuclear physics near this UV fixed point??



$$a_s^{-1} \sim \frac{m_\pi - m_\pi^*}{m_\pi} \Lambda_{QCD}$$

So what???

All we've done is reproduce ERT!

Let's consider *electroweak probes*

S-wave NN primer

$$\text{N}: \quad I = \frac{1}{2}\ ,\ S = \frac{1}{2}$$

$$\text{NN}: \quad \left(\frac{1}{2} \otimes \frac{1}{2}\right)_{\text{I}} = \left(0^{\text{A}} \otimes 1^{\text{S}}\right)_{\text{I}}$$

$$\left(\frac{1}{2} \otimes \frac{1}{2}\right)_{\text{S}} = \left(0^{\text{A}} \otimes 1^{\text{S}}\right)_{\text{S}}$$

Antisymmetric WF:

spin-isospin	spectroscopic	"field"
$S = 1 \quad , \quad I = 0$	$^3S_1$	t_k *deuteron*
$S = 0 \quad , \quad I = 1$	$^1S_0$	s_a

$$|\,d\,\rangle \rightarrow t$$

$$|\,np\,\rangle_s \rightarrow s_3$$

$$|\,nn\,\rangle_s \rightarrow \frac{1}{\sqrt{2}}(s_1 + is_2)$$

$$|\,pp\,\rangle_s \rightarrow \frac{1}{\sqrt{2}}(s_1 - is_2)$$

How do we treat the deuteron *bound state* in EFT?

Recall ERT in $^3S_1$

$$\mathcal{A}_2(p) = \frac{4\pi}{M}\frac{1}{-\frac{1}{a}+\frac{1}{2}r_0p^2+\ldots-\mathrm{i}p}$$

$$-\frac{1}{a}+\frac{r_0}{2}(p^*)^2-ip^*=0$$

Binding momentum: $\gamma = -ip^*$

Binding energy: $B = \frac{\gamma^2}{M} = 2.224575(9)\ \mathrm{MeV}$

Deuteron source method

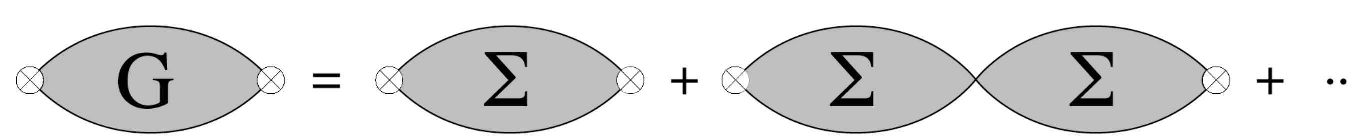
Interpolating field: $\otimes$ **=** $\mathcal{D}_i \equiv N^T P_i N$ $\qquad P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\,\tau_2$ $\qquad ({}^3S_1)$

$$G(\overline{E})\,\delta_{ij} = \int \mathrm{d}^4x\, e^{-i(Et-\mathbf{p}\cdot\mathbf{x})}\,\langle 0|\mathrm{T}\left[\mathcal{D}_i^\dagger(x)\mathcal{D}_j(0)\right]|0\rangle = \delta_{ij}\frac{i\mathcal{Z}(\overline{E})}{\overline{E}+B+i\varepsilon}$$

$$\overline{E} \equiv E - \frac{\mathbf{p}^2}{4M} + \ldots \qquad\qquad E \equiv (p^0 - 2M)$$

WF renormalization: $\mathcal{Z}(-B) \equiv Z = -i\left[\frac{\mathrm{d}G^{-1}(\overline{E})}{\mathrm{d}E}\right]^{-1}_{\overline{E}=-B}$

$$G = \frac{\Sigma}{1+iC_0\Sigma}$$ **=**

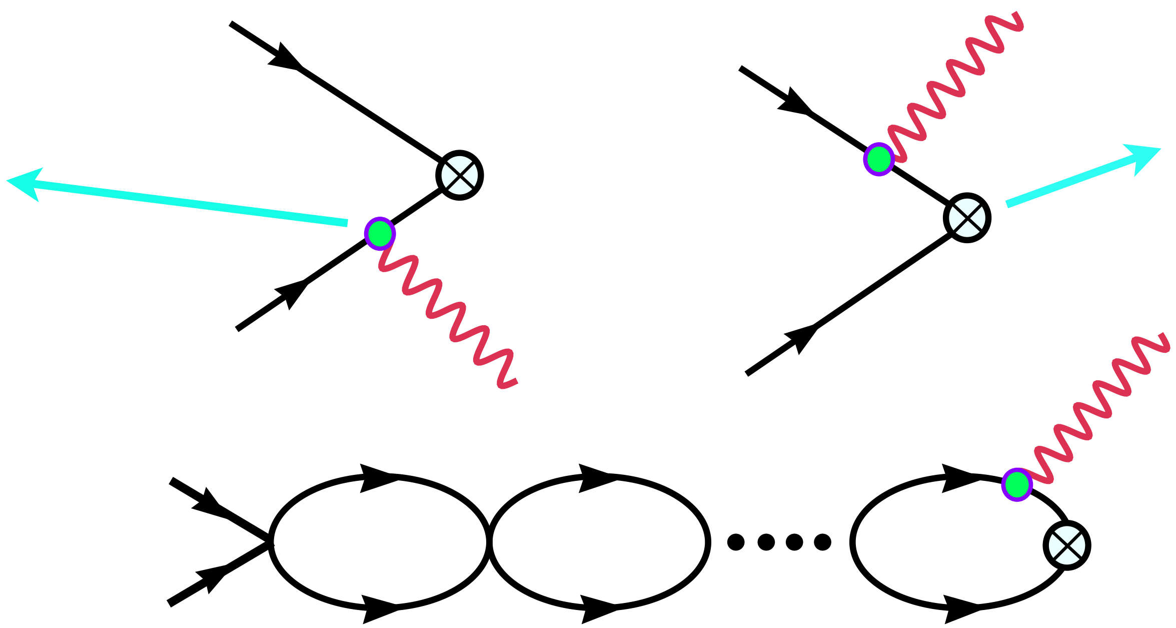


Example: $n\,p \to d\,\gamma$

$$^3S_1 \;:\quad P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\;\tau_2 \qquad\qquad ^1S_0 \;:\quad \overline{P}_i = \frac{1}{\sqrt{8}}\sigma_2\;\tau_2\tau_i$$

LO

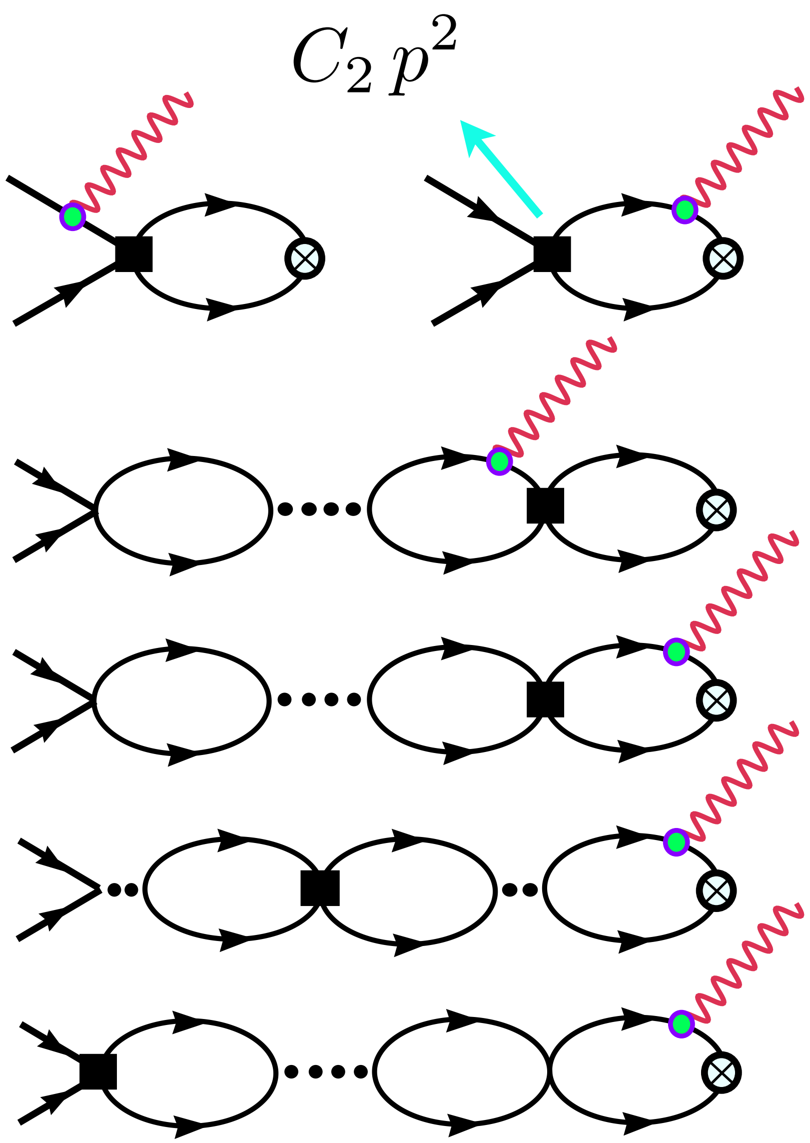
$$\frac{e}{2M_N}N^\dagger(\kappa_0+\kappa_1\tau_3)\boldsymbol{\sigma}\cdot\mathbf{B}N$$



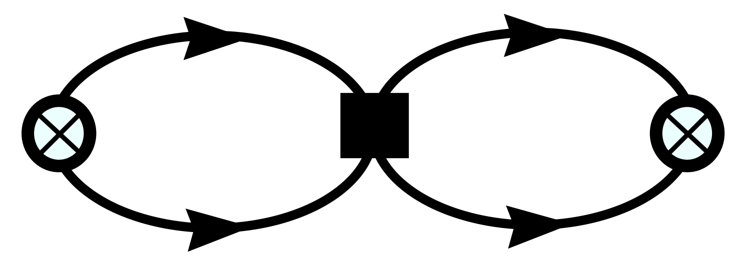
$$\mathcal{D}_i(x) \equiv N^T P_i N(x)$$

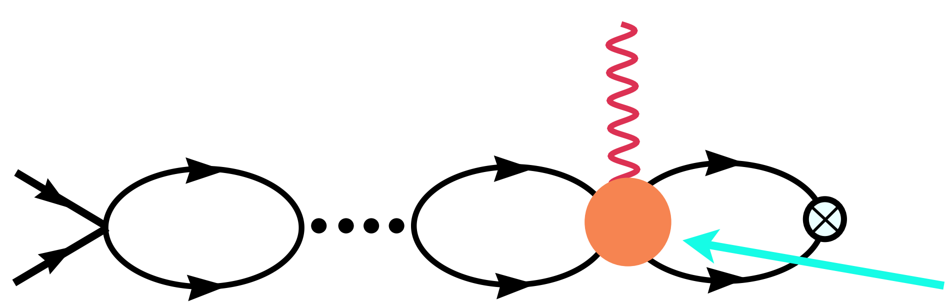
$$\sigma = \frac{8\pi\alpha\gamma^5\kappa_1^2 a_0^2}{vM_N^5}\left(1-\frac{1}{\gamma a_0}\right)^2$$

NLO



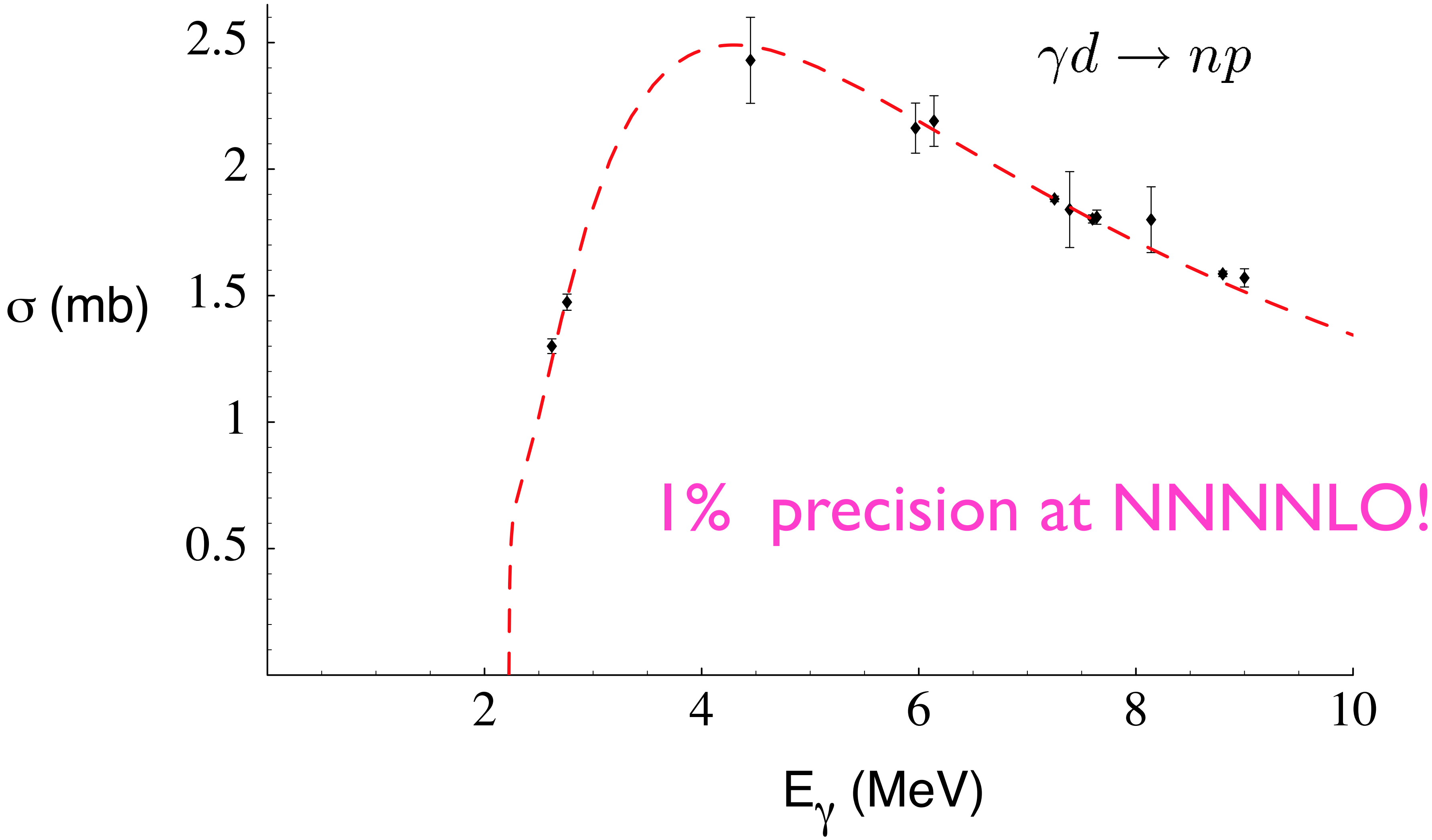
WF renormalization:





$$e \not{L}_1 \;\left(N^T P_i N\right)^\dagger \left(N^T \overline{P}_3 N\right) \mathbf{B}_i$$

Beyond ERT!



Also achieved by model calculations, however *EFT* provides:

A *systematic procedure* for computing corrections to the desired accuracy in the most economical way.

Dibaryon (Dimer) method

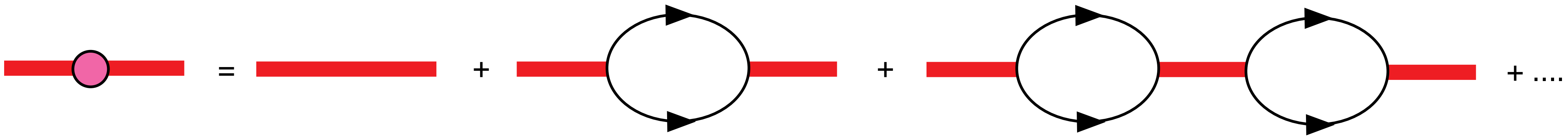
$$^3S_1 \;: \qquad P_i = \frac{1}{\sqrt{8}}\sigma_2\sigma_i\;\tau_2 \qquad\qquad {}^1S_0 \;: \qquad \overline{P}_i = \frac{1}{\sqrt{8}}\sigma_2\;\tau_2\tau_i$$

$$\mathcal{L}_s = \quad -s_a^\dagger\left[iv\cdot D + \frac{1}{4M}[(v\cdot D)^2 - D^2] + \Delta_s\right]s_a - y_s\left[s_a^\dagger(N^T\overline{P}_a N) + \text{h.c.}\right]$$

$$\mathcal{L}_t = \quad -t_i^\dagger\left[iv\cdot D + \frac{1}{4M}[(v\cdot D)^2 - D^2] + \Delta_t\right]t_i - y_t\left[t_i^\dagger(N^T P_i N) + \text{h.c.}\right]$$

$$D_\mu = \partial_\mu - i\mathcal{V}^{ext}{}_\mu \qquad\qquad \Delta_{s,t} = \frac{2}{Mr_{s,t}}\left(\frac{1}{a} - \mu\right)$$

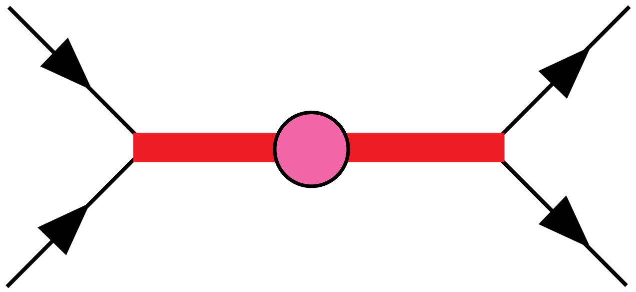
$$y_s = \frac{2}{M}\sqrt{\frac{2\pi}{r_0^s}} \qquad\qquad y_t = \frac{2}{M}\sqrt{\frac{2\pi}{r_0^t}}$$



Dressed deuteron propagator

$$D(\overline{E}) = \frac{4\pi}{My^2} \frac{i}{\mu + \frac{4\pi}{My^2}\Delta - \frac{4\pi}{My^2}\overline{E} + i\sqrt{M\overline{E}}}$$

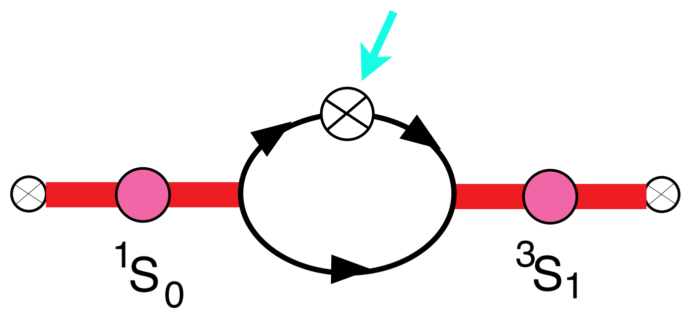
$$-i\mathcal{A}_2(\overline{E}) = y\, D(\overline{E})\, y =$$

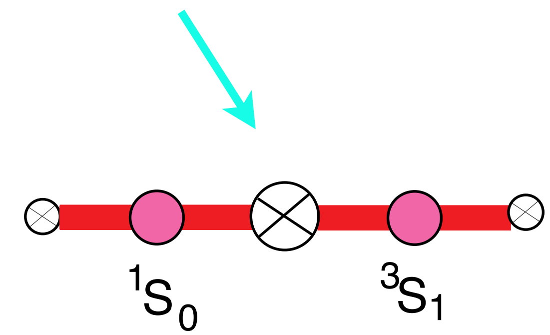


Recovers ERT!

Example: deuteron *weak* processes

$$\mathcal{A}_i^a = -\left[\left(\frac{r_0^s + r_0^t}{2\sqrt{r_0^s r_0^t}}\right) g_A + \frac{l_{1A}}{M\sqrt{r_0^s r_0^t}}\right]\left[s_a^\dagger t_i + \text{h.c.}\right]$$





ν-d

$$\mathcal{L}^{CC} = -\frac{G_F}{\sqrt{2}} l_+^\mu J_\mu^- + h.c.$$

$$l_+^\mu = \overline{\nu}\gamma^\mu(1-\gamma_5)e$$

$\overline{\nu}$-d

$$\mathcal{L}^{NC} = -\frac{G_F}{\sqrt{2}} l_Z^\mu J_\mu^Z$$

$$l_Z^\mu = \overline{\nu}\gamma^\mu(1-\gamma_5)\nu$$

$$J_\mu^- = (\mathcal{V}_\mu^1 - \mathcal{A}_\mu^1) - i(\mathcal{V}_\mu^2 - \mathcal{A}_\mu^2)$$

$$J_\mu^Z = -2\sin^2\theta_W \mathcal{V}_\mu^S + (1 - 2\sin^2\theta_W)\mathcal{V}_\mu^3 - \mathcal{A}_\mu^S - \mathcal{A}_\mu^3$$

Sudbury Neutrino Observatory (SNO)

^8B solar flux $\longrightarrow$

$$\begin{array}{lcll} \nu_e + d & \rightarrow & p + p + e^- & \text{(CC)} \\ \nu_x + d & \rightarrow & p + n + \nu_x & \text{(NC)} \\ \nu_x + e^- & \rightarrow & \nu_x + e^- & \text{(ES)} \end{array}$$

$x = e, \mu, \tau$

all active neutrino components of solar flux

$E_{\nu,\overline{\nu}} = 10$ MeV $\qquad \left(10^{-42}\ \text{cm}^2\right)$

$$\sigma(\nu_e d \rightarrow e^- pp) = 4.07 + 0.12\ l_{1,A}$$
$$\sigma(\nu_x d \rightarrow \nu_x np) = 1.76 + 0.056\ l_{1,A}$$
$$\sigma(\overline{\nu}_x d \rightarrow \overline{\nu}_x np) = 1.66 + 0.052\ l_{1,A}$$
$$\sigma(\overline{\nu}_e d \rightarrow e^+ nn) = 1.93 + 0.059\ l_{1,A}$$

$\longrightarrow$ $l_{1,A}$ uncertainty

% level in σ

$l_{1,A}$ will be measured in $\mu^- + d \rightarrow \nu_\mu + n + n$

Lecture IV: χ-PT Primer

- QCD and chiral symmetry
- Chiral perturbation theory
- Power counting
- QCD in finite volume
- Symanzik action
- $\pi\pi$

Why is QCD interesting?

- “Background” for beyond-the-Standard-Model physics
- Hadronic/Nuclear mysteries (S-wave NN scattering lengths)
- No experiments (ΛN, $\pi\pi$, $K\pi$, KK)
- Quark-mass dependence (lattice QCD)
- Because the Nobel committee says so

$$\mathcal{L}_{\text{QCD}} = \sum_{i=1}^{3} \left(\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i \right) - \frac{1}{2} \text{Tr} G_{\mu\nu} G^{\mu\nu}$$

u,d,s active flavors

$$D_\mu = \partial_\mu + ig A_\mu \qquad A_\mu = A_\mu^a T_a$$

$$T_a \in SU(3)$$

$$\sum_i \bar{q}_i i \not{D} q_i = \sum_i \left(\bar{q}_{Li} i \not{D} q_{Li} + \bar{q}_{Ri} i \not{D} q_{Ri} \right)$$

$$q_{L,R} = \tfrac{1}{2}(1 \pm \gamma_5) q$$

$U(3)_L \times U(3)_R$ invariance

$U(1)_A$ *anomalous*

$$U(1)_V \times SU(3)_L \times SU(3)_R$$

Baryon number

Chiral symmetry:

$$q_{Li} \to L_{ij} q_{Lj} \qquad q_{Rj} \to R_{ij} q_{Rj}$$

$$\sum_i m_i \bar{q}_i q_i = \sum_{i,j} \bar{q}_{Ri} M_{ij} q_{Lj} + h.c.$$

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

If mass matrix were field with:

$$M \to R M L^\dagger$$

spurion

then mass term would be a chiral invariant

NOTE

- $m_u\,,\, m_d \ll m_s$ $\quad SU(2)_L \times SU(2)_R$ better than $SU(3)_L \times SU(3)_R$
- $m_u = m_d = m_s \neq 0$ $\quad SU(3)_V \quad (L = R)$ is exact
- $m_u = m_d \neq 0$ $\quad SU(2)_V$ *Isospin* $(L = R)$ is exact
- $3 \times U(1) \leftrightarrow B + I_3 + Y$

Consequences of chiral symmetry?


Assume ground state baryon octet of positive parity $\mathcal{P}$ *:*

$$|B\rangle \sim |(1,8)\rangle \;+\; |(8,1)\rangle$$

$$\mathcal{P}|(L,R)\rangle \;=\; |(R,L)\rangle$$

$$\mathcal{P}|B\rangle \;=\; |B\rangle$$

Must also have:

$$|B^*\rangle \sim |(1,8)\rangle \;-\; |(8,1)\rangle$$

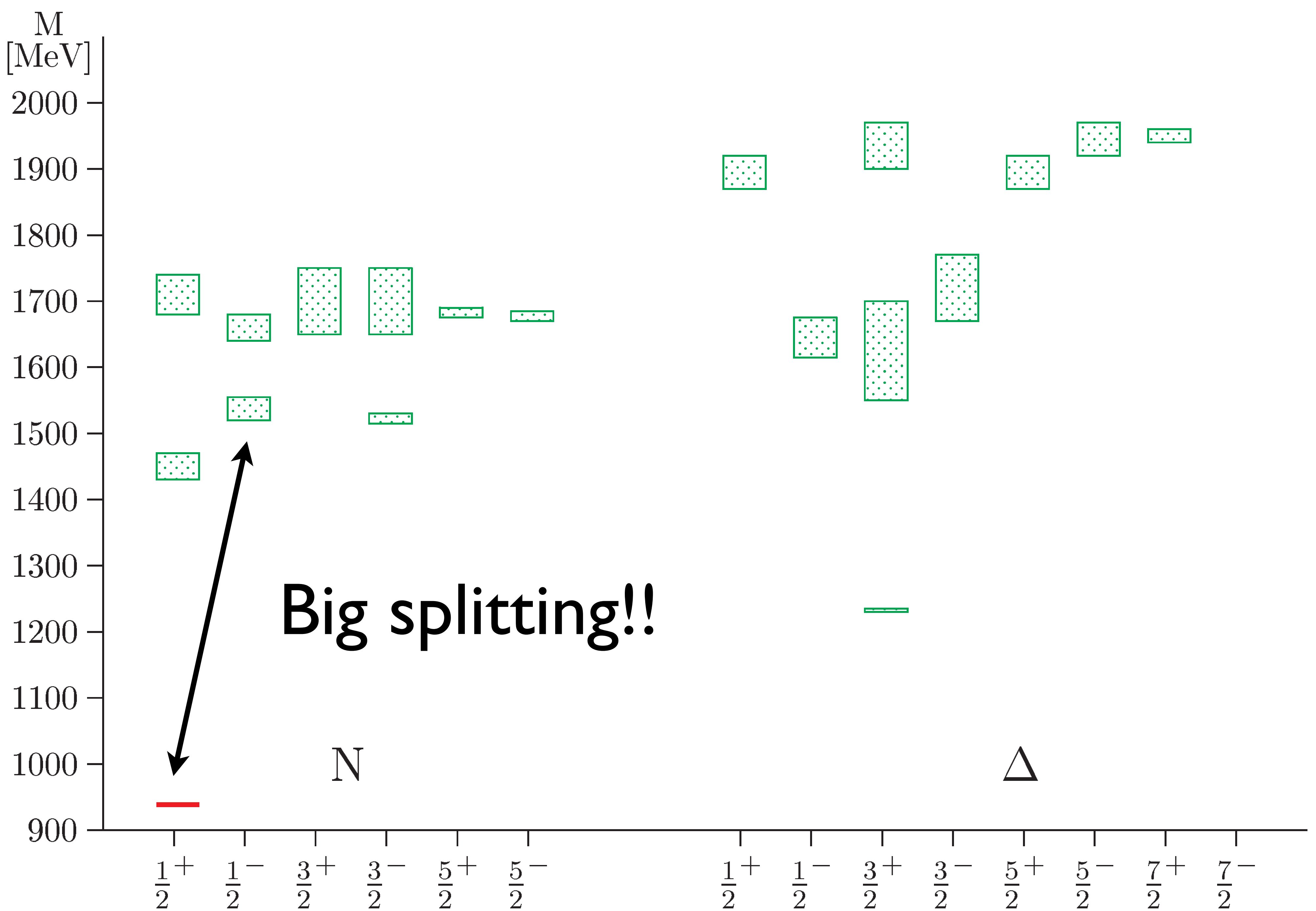
$$\mathcal{P}|B^*\rangle \;=\; -|B^*\rangle$$

$$\mathcal{H}_{QCD} \in (1,1)$$

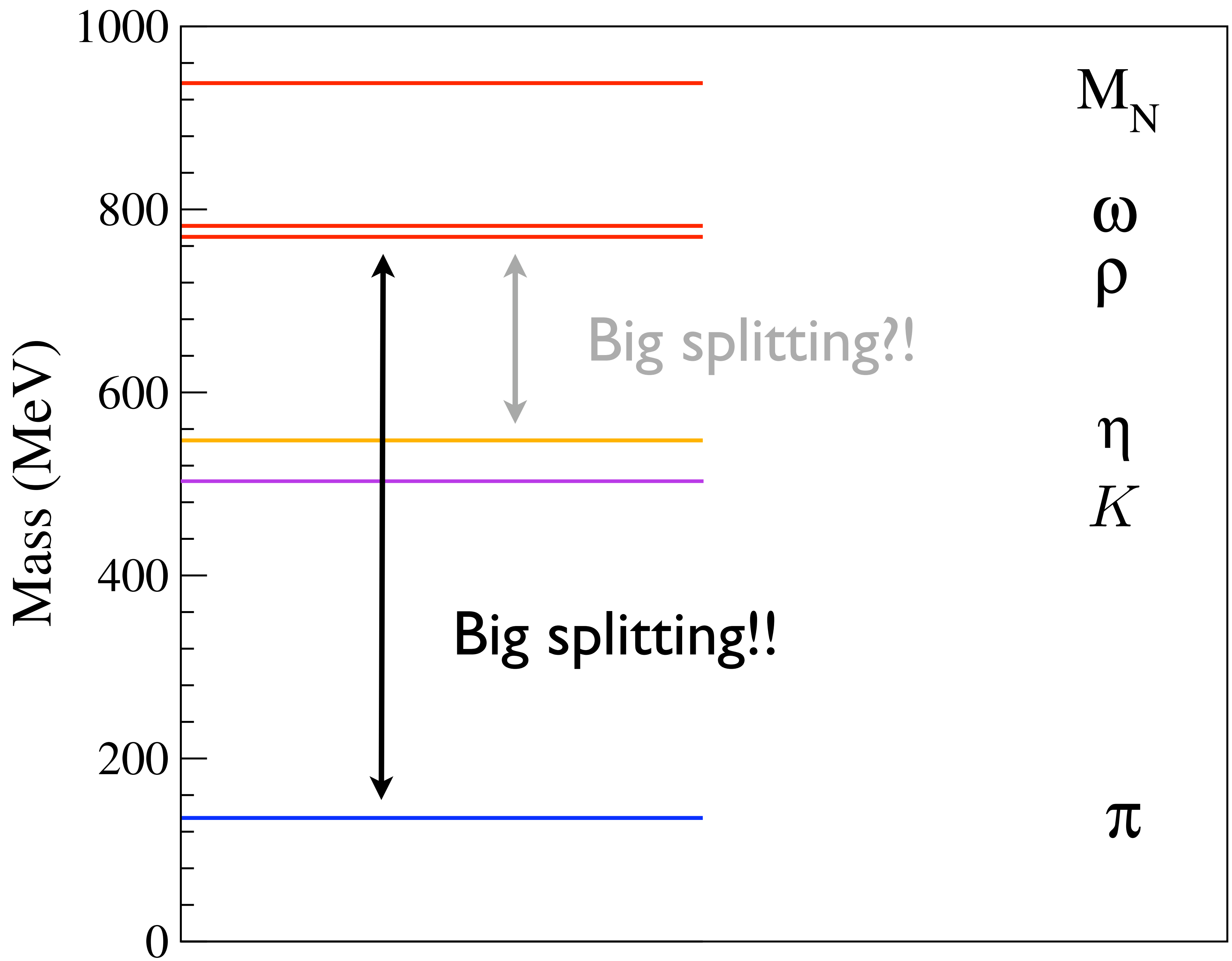
$$M_B \;=\; \langle B|\mathcal{H}_{QCD}|B\rangle \;=\; \langle B^*|\mathcal{H}_{QCD}|B^*\rangle \;=\; M_{B^*}$$

Parity doubling! (Wigner-Weyl)

EXPERIMENT: Baryons



EXPERIMENT: Mesons



$$G \;=\; SU(3)_L \;\times\; SU(3)_R$$

Wigner-Weyl realization of G
ground state is symmetric

$$\langle 0|\,\bar{q}_{\mathrm{R}}\, q_{\mathrm{L}}\,|0\rangle = 0$$

ordinary symmetry
spectrum contains parity partners
degenerate multiplets of G

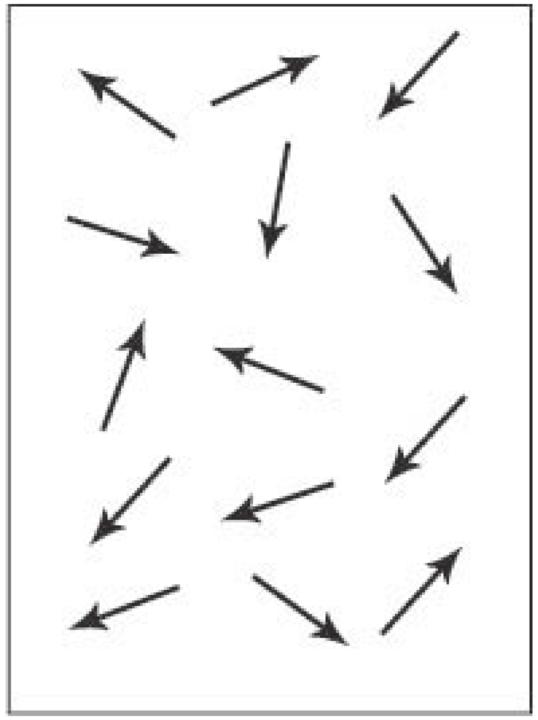
Nambu-Goldstone realization of G
ground state is asymmetric

$$\langle 0|\,\bar{q}_{\mathrm{R}}\, q_{\mathrm{L}}\,|0\rangle \neq 0$$

"order parameter"
spontaneously broken symmetry
spectrum contains Goldstone bosons
degenerate multiplets of $SU(3)_V \subset G$

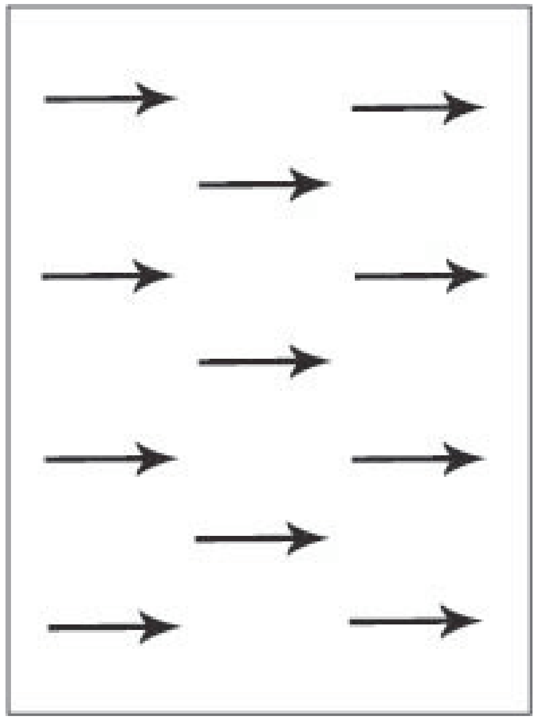
Analogy

Above T_c:



$\langle \mathbf{M} \rangle = 0$

Below T_c:



$\langle \mathbf{M} \rangle \neq 0$

Ferromagnetism	**QCD**
Ground state $\vert\mathrm{magnet}\rangle$	QCD vacuum $\vert 0\rangle$
$\langle \mathrm{magnet} \vert \mathbf{M} \vert \mathrm{magnet} \rangle$	$\langle 0 \vert \bar{q} q \vert 0 \rangle$
$O(3)$	$SU(2)_A$
Low temperature	Low energy, also T
Magnons	Pions

Assume:

$$\langle 0|\bar{q}_{Rj} q_{Li}|0\rangle = \Lambda^3 \delta_{ij}$$

$$SU(3)_L \times SU(3)_R \to SU(3)_V$$

$$q_{Li} \to L_{ij} q_{Lj} \qquad q_{Rj} \to R_{ij} q_{Rj}$$

$$\delta_{ij} \to (LR^\dagger)_{ij} \equiv \Sigma_{ij}$$

mass scale t.b.d

Excitations of the condensate:

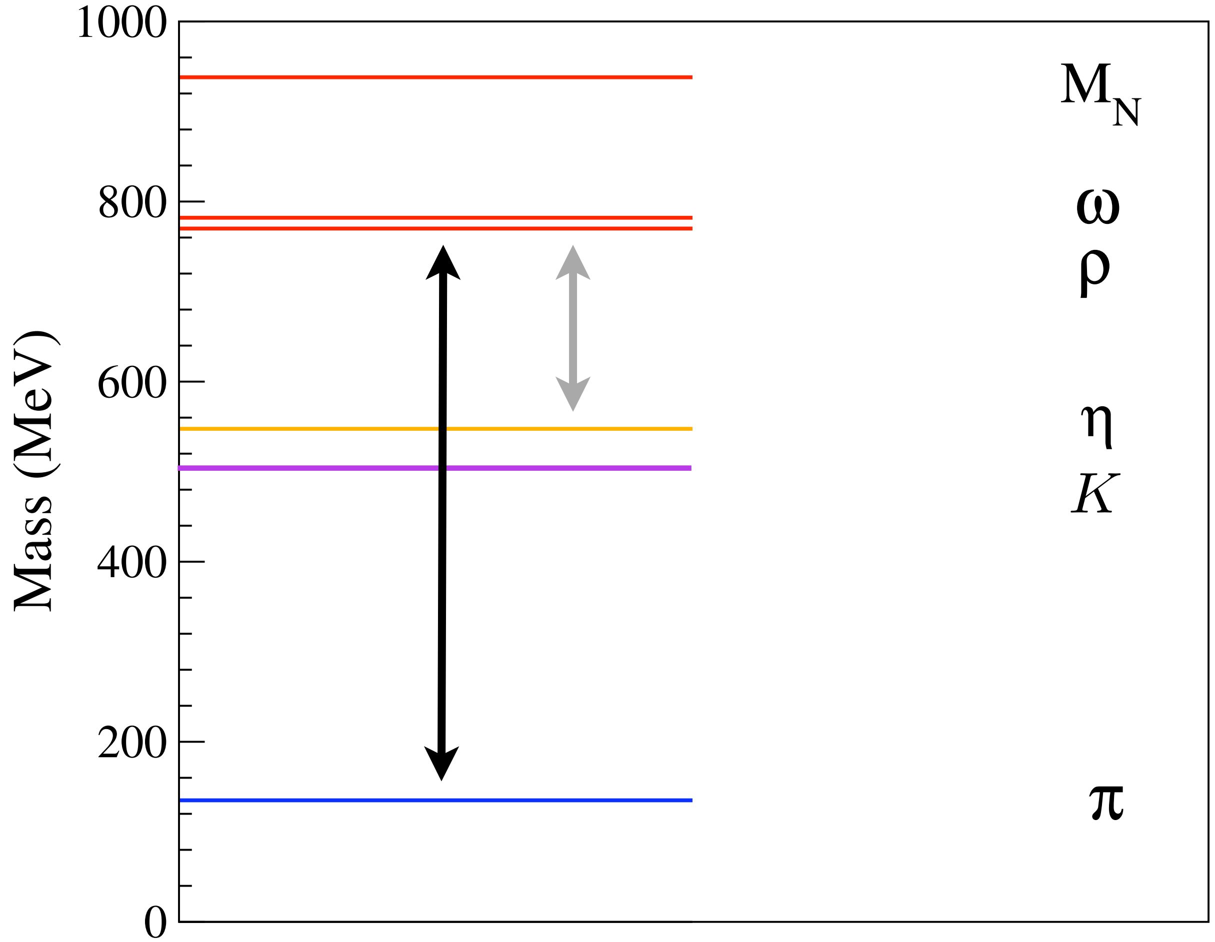
$$\Sigma \to \Sigma(x) \equiv e^{2i\boldsymbol{\pi}(x)/f}$$

$$\boldsymbol{\pi}(x) = \pi_a(x) T_a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \sim \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}$$

The Goldstone bosons

Chiral Perturbation Theory $(\chi - \mathrm{PT})$

EFT that encodes the interactions of the Goldstone modes with themselves and with other hadronic degrees of freedom.



Λ_χ

$SU(3)_L \times SU(3)_R$

$$\frac{(p\,,\,m_\pi\,,\,m_K\,,\,m_\eta)}{\Lambda_\chi} \ll 1$$

$SU(2)_L \times SU(2)_R$

$$\frac{(p\,,\,m_\pi)}{\Lambda_\chi} \ll 1$$

$$SU(3)_L \times SU(3)_R \quad : \quad \Sigma \longrightarrow L\Sigma R^\dagger$$

$$\mathcal{L}_0 \;=\; \frac{f^2}{4}\,\mathrm{Tr}\partial_\mu\Sigma^\dagger\partial_\mu\Sigma = \frac{1}{2}\partial_\mu\pi_a\partial_\mu\pi_a \;+\; interactions$$

normalized kinetic term

$$\langle 0|\bar{u}\gamma^\mu(1-\gamma_5)d|\pi^-(p)\rangle \equiv i\sqrt{2}\,f_\pi p^\mu \qquad f_\pi \;=\; 92.4 \pm 0.25\ \mathrm{MeV}$$

$\pi \longrightarrow \mu\nu$

$$||$$

$$2\left(j^\mu_{L1} + i j^\mu_{L2}\right)$$

$$\chi - \mathrm{PT}: \quad j^\mu_{La} = -i\frac{f^2}{2}\,\mathrm{Tr}T_a\Sigma^\dagger\partial^\mu\Sigma = \frac{1}{2}\,f\partial^\mu\pi_a + \dots$$

Matching: $$f = f_\pi = 93\ \mathrm{MeV}$$

Explicit chiral symmetry breaking:

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \neq 0$$

$$SU(3)_L \times SU(3)_R \quad : \quad M \to R M L^\dagger$$ spurion

$$\mathcal{L}_M = \lambda_M \frac{f^2}{2} \left(\mathrm{Tr} M \Sigma + \mathrm{h.c.}\right) = \frac{1}{2} m_{\pi_a}^2 \pi_a \pi_a + \textit{interactions}$$

λ_M: mass scale t.b.d

$$m_\pi^2 = \lambda_M (m_u + m_d) \qquad m_{K^+}^2 = \lambda_M (m_u + m_s) \qquad m_{K^0}^2 = \lambda_M (m_d + m_s)$$

$$m_\eta^2 = \frac{1}{3} \lambda_M (m_u + m_d + 4 m_s) + \mathcal{O}\left((m_u - m_d)^2\right)$$

$$m_{\pi^0}^2 = \lambda_M (m_u + m_d) + \mathcal{O}\left((m_u - m_d)^2\right)$$

$$3 m_\eta^2 + m_\pi^2 = 4 m_K^2$$

Gell-Mann Okubo

EXP: 1%

$$\lambda_M \;??$$

$$\mathcal{H}_{QCD} = \ldots + \sum_i m_i \overline{q}_i q_i + \ldots$$

$$\langle 0|\overline{q}_i q_i|0\rangle = \frac{\partial}{\partial m_i}\langle 0|\mathcal{H}_{QCD}|0\rangle = \frac{\partial \mathcal{E}_0}{\partial m_i}$$

→ vaccum energy density

$\chi - \mathrm{PT}:$
$$\mathcal{E}_0 = constant - \lambda_M \frac{f^2}{2}(\mathrm{Tr} M\Sigma + \mathrm{h.c.}) + \ldots$$

Matching: $$\langle 0|\overline{q}_i q_i|0\rangle = \lambda_M f^2$$

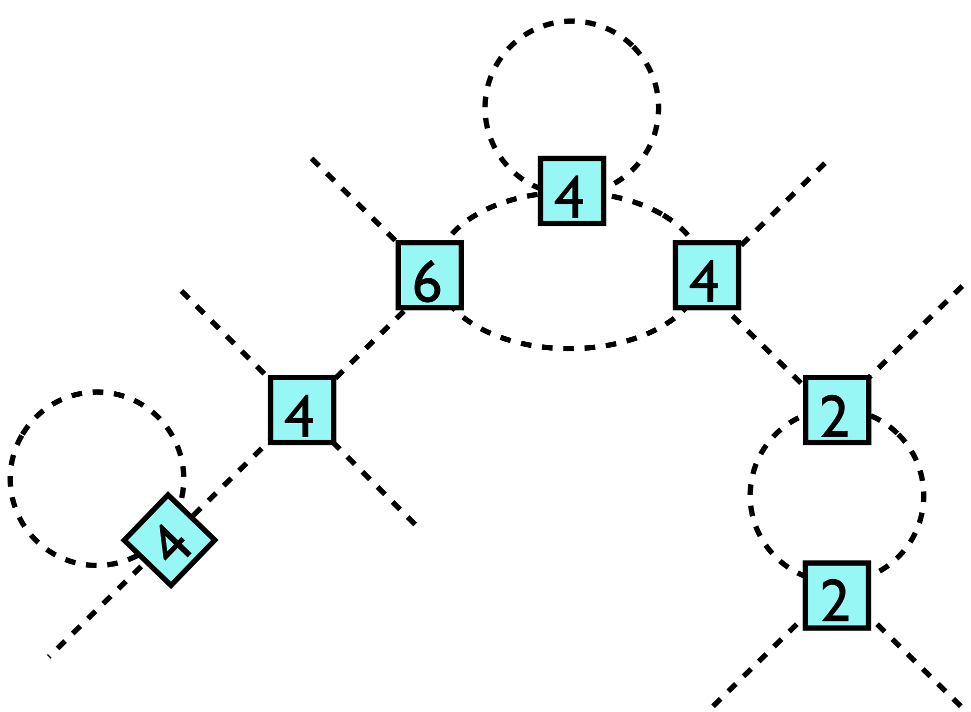
$$m_\pi^2 = \frac{\langle 0|\overline{u}u|0\rangle}{f^2}(m_u + m_d)$$

Gell-Mann-Oakes-Renner

Power counting

Consider generic connected diagram:

V_d	:	vertices with d derivatives
E	:	external lines
I	:	internal lines
L	:	loops



note: $p = (q, m_\pi)$

$$p^{\sum_d dV_d - 2I + 4L}$$

+

$$L = I - \sum_d V_d + 1$$

↓

$$N = \sum_d \frac{1}{2}(d-2)V_d + L$$

$$f^2 p^2 \left(\frac{p^2}{f^2}\right)^N \left(\frac{1}{f}\right)^E$$

Generic *power counting*

Sub-leading Lagrangian

$SU(3)_L \times SU(3)_R$

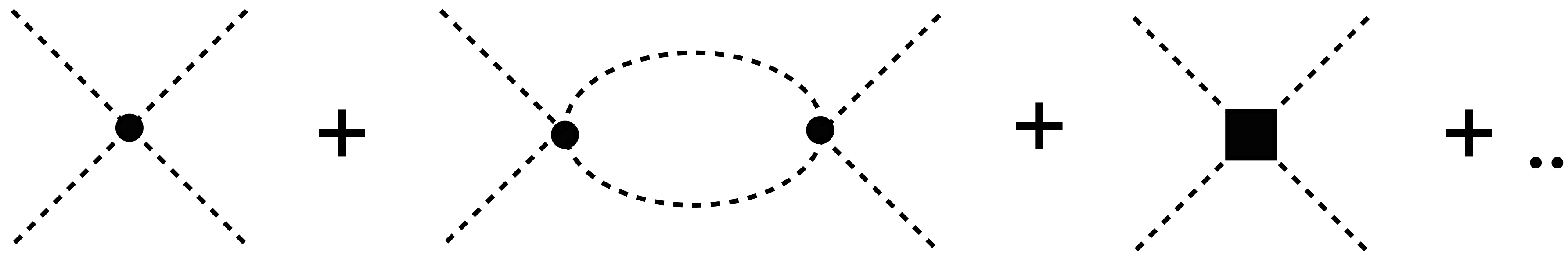
$$
\begin{aligned}
\mathcal{L}_{p^4} = & \, L_1 \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + L_2 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right) \, \mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma\right) \\
& + L_3 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma \partial_\nu \Sigma^\dagger \partial^\nu \Sigma\right) + L_4 \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right) \\
& + L_5 \mathrm{Tr}\left(\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\left(\chi \Sigma + \mathrm{h.c.}\right)\right) + L_6 \left(\mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right)\right)^2 \\
& + L_7 \left(\mathrm{Tr}\left(\chi \Sigma - \mathrm{h.c.}\right)\right)^2 + L_8 \mathrm{Tr}\left(\chi \Sigma \chi \Sigma + \mathrm{h.c.}\right)
\end{aligned}
$$

$SU(2)_L \times SU(2)_R$

$$
\begin{aligned}
\mathcal{L}_{p^4} = & \, \frac{\ell_1}{4} \left(\mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right)\right)^2 + \frac{\ell_2}{4} \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial_\nu \Sigma\right) \, \mathrm{Tr}\left(\partial^\mu \Sigma^\dagger \partial^\nu \Sigma\right) \\
& + \frac{\ell_4}{4} \mathrm{Tr}\left(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right) \mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right) + \frac{(\ell_3 + \ell_4)}{4} \left(\mathrm{Tr}\left(\chi \Sigma + \mathrm{h.c.}\right)\right)^2 + \frac{\ell_7}{4} \left(\mathrm{Tr}\left(\chi \Sigma - \mathrm{h.c.}\right)\right)^2
\end{aligned}
$$

$$\chi \equiv 2 \lambda_M M$$

Example: $\pi\pi$ scattering



$$f^2 p^2 \left(\frac{p^2}{f^2}\right)^N \left(\frac{1}{f}\right)^E$$

$$\frac{p^2}{f^2} \qquad \frac{p^4}{f^4}\log\frac{p^2}{\mu^2} \qquad \frac{p^4}{f^4}$$

$I = 2$

s-wave scattering length

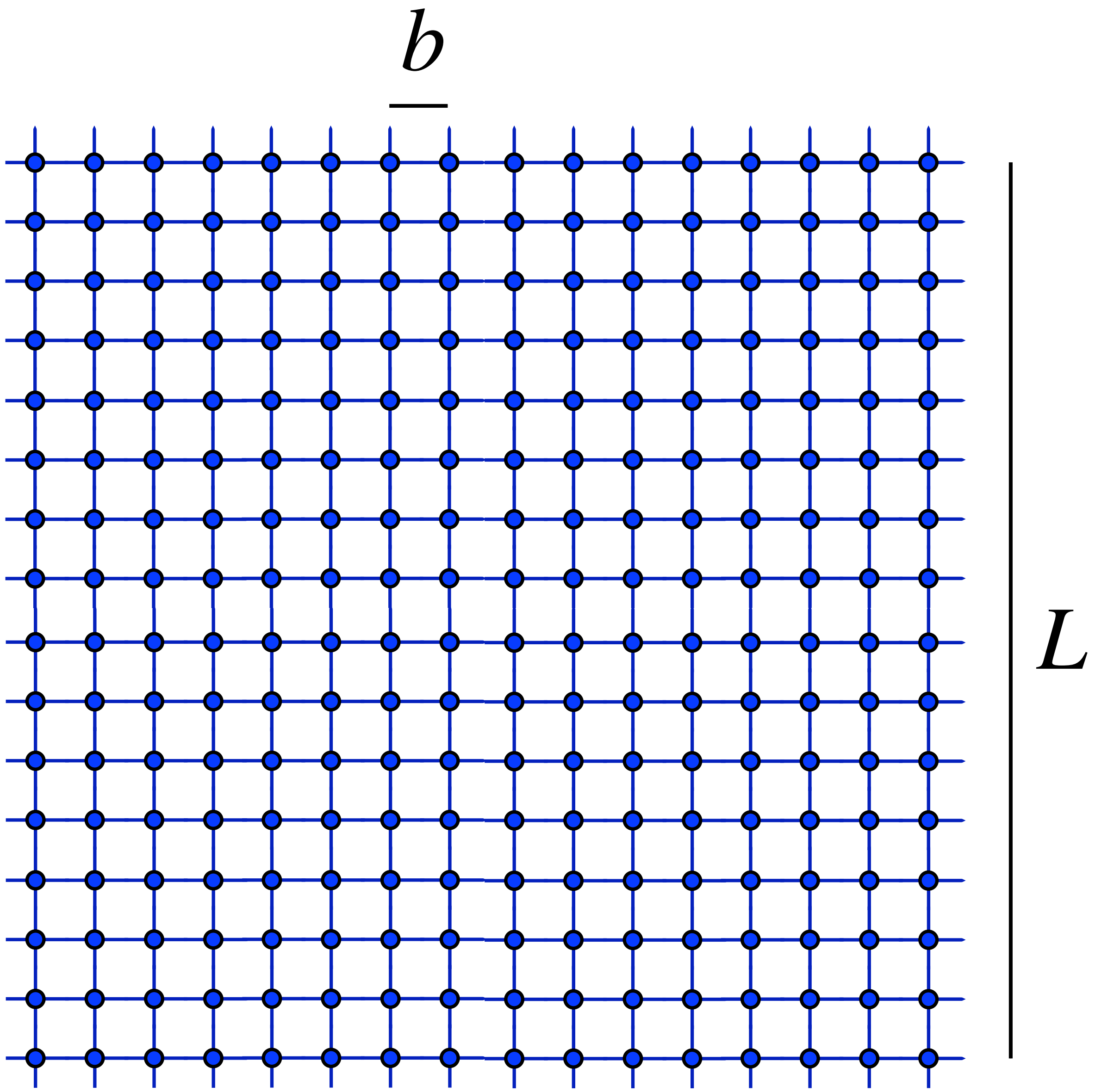
$$m_\pi a_2 = -\frac{m_\pi^2}{8\pi f_\pi^2}\left[1 + \frac{3m_\pi^2}{16\pi^2 f_\pi^2}\left(\log\frac{m_\pi^2}{\mu^2} + \bar{l}_{\pi\pi}(\mu)\right)\right]$$

$$\bar{l}_{\pi\pi} \equiv -\frac{64\pi^2}{3}\left[4(\ell_1+\ell_2)+\ell_3-\ell_4\right]+1$$

Not constrained by symmetries!

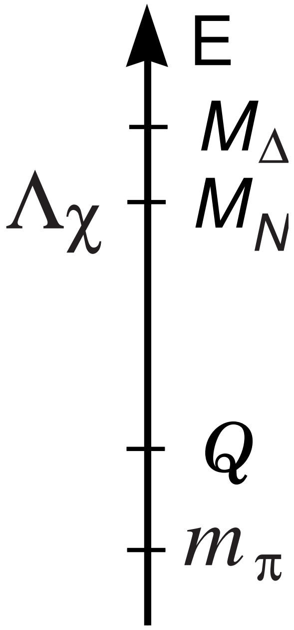
But related to other processes!

Lattice QCD



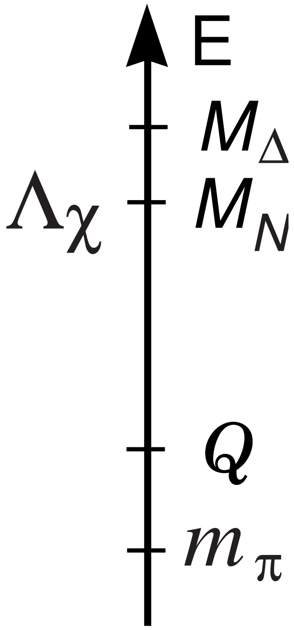
$$\mathrm{COST} \sim (L)^4\ (b)^{-6.5}\ (M_q)^{-2.5}$$

QCD:



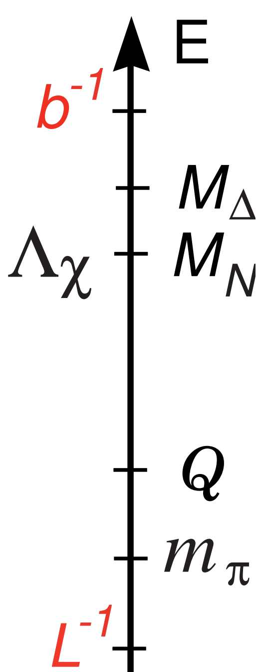
$$\frac{Q}{\Lambda_\chi}, \quad \frac{m_\pi}{\Lambda_\chi}, \quad \frac{M_\Delta - M_N}{\Lambda_\chi}, \; \cdots$$

QCD:



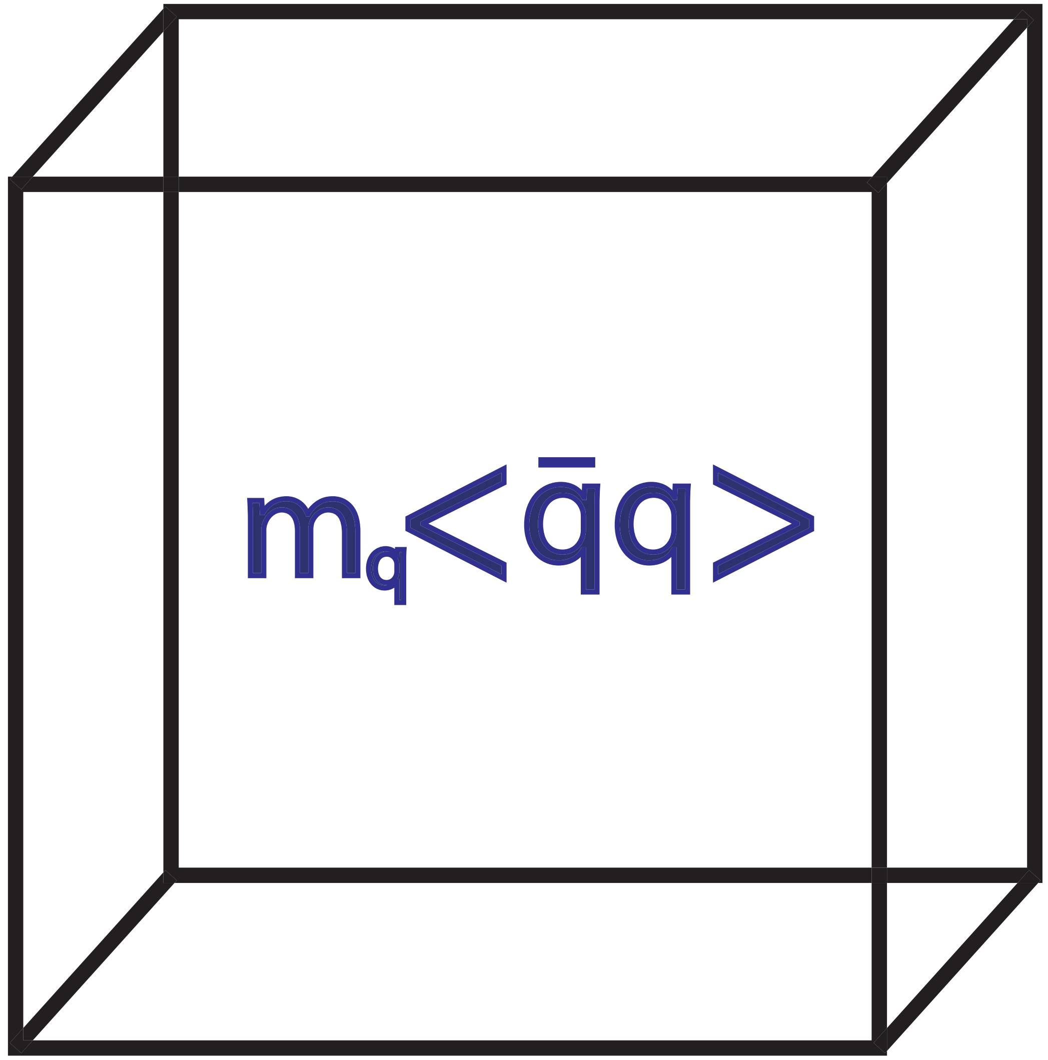
$$\frac{Q}{\Lambda_\chi}, \quad \frac{m_\pi}{\Lambda_\chi}, \quad \frac{M_\Delta - M_N}{\Lambda_\chi}, \ \ldots$$

Lattice QCD :



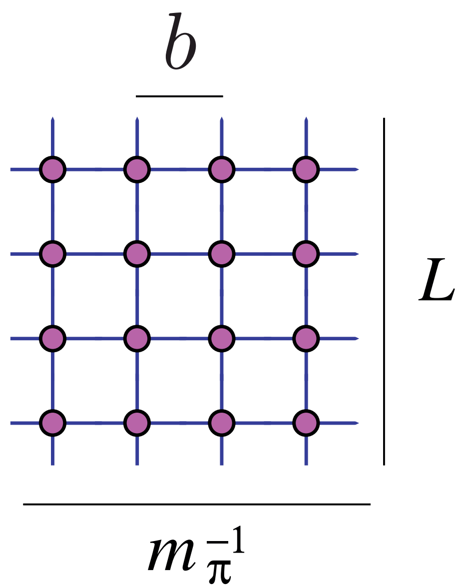
$$b\, m_\pi, \quad e^{-m_\pi L}, \quad m_\pi L, \quad \frac{1}{L\Lambda_\chi}, \ \ldots$$

What happens to chiral symmetry breaking at *finite V*?



$$|\mathbf{p}| = \frac{2\pi|\mathbf{n}|}{L} \ll \Lambda_\chi \qquad \Longrightarrow \qquad fL \gg 1$$

- $m_q\langle\bar{q}q\rangle L^4 = (m_\pi L)^2(fL)^2 \sim p^{-2} \gg 1$:

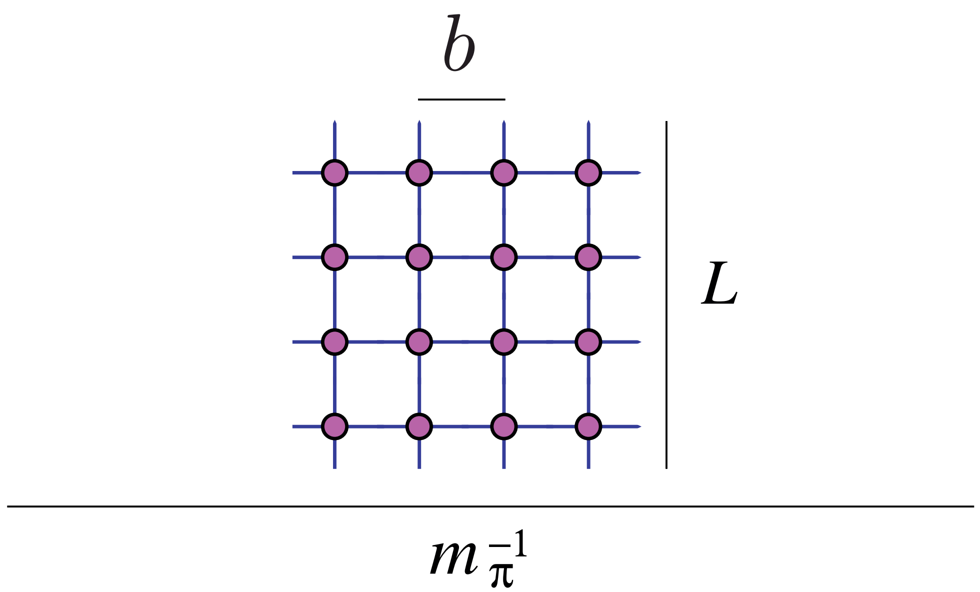


$m_\pi L \gtrsim 1$ <u>p regime</u>

$$L^{-1} \sim m_\pi \sim p$$

$$(L_t \gg L_s) \quad \Longrightarrow \quad \int d^4l \to \int dl_0 \sum_{\vec{l}}$$

- $(m_\pi L)^2(fL)^2 \sim \epsilon^0 \lesssim 1$:



$m_\pi L \ll 1$ <u>ϵ regime</u>

$$L^{-1} \sim \sqrt{m_\pi} \sim \epsilon$$

Momentum zero-modes nonperturbative

What happens to chiral symmetry breaking at *finite b*?

Symanzik action:

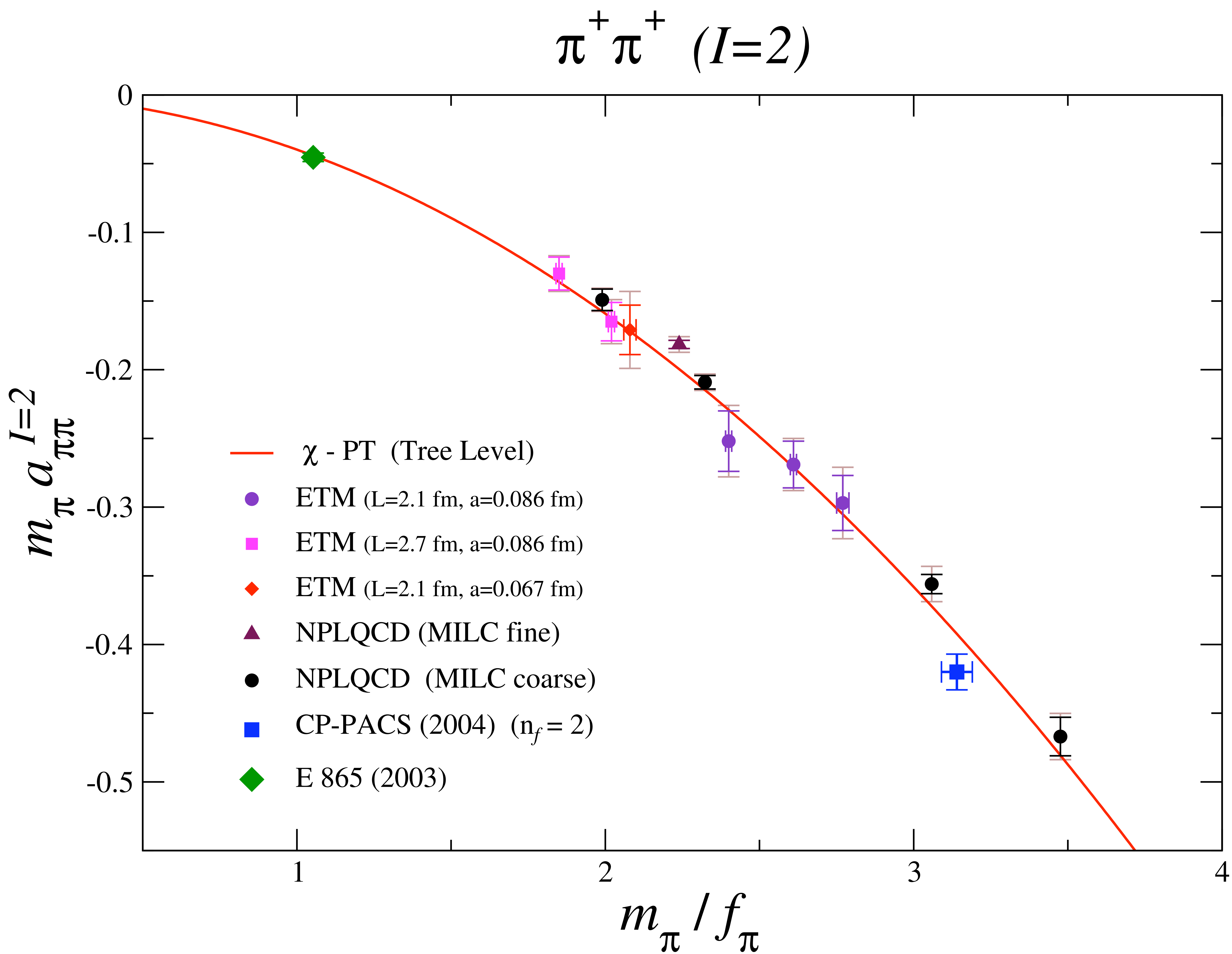
$$\mathcal{O}(b): \quad \mathcal{L}_{\mathrm{QCD}}^{\mathrm{EFT}} = \sum_{i=1}^{3} \left(\bar{q}_i i \not{D} q_i - m_i \bar{q}_i q_i\right) - \frac{1}{2} \mathrm{Tr} G_{\mu\nu} G^{\mu\nu} + b\, c_{sw} \sum_i \bar{q}_i \sigma_{\mu\nu} G^{\mu\nu} q_i \; + \; \ldots$$

Sheikholeslami-Wohlert

$$A \rightarrow R A L^\dagger$$

$$\mathcal{L}_{M,A} = \lambda_M \frac{f^2}{2} \left(\mathrm{Tr} M \Sigma + \mathrm{h.c.}\right) \; + \; \lambda_A \frac{f^2}{2} \left(\mathrm{Tr} A \Sigma + \mathrm{h.c.}\right)$$

$$m_\pi^2 \quad = \quad \lambda_M \; (m_u + m_d) \; + \; 2\lambda_A \; b \; c_{sw}^{(V)}$$



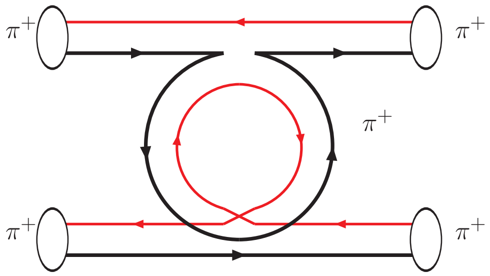
$\pi^+\pi^+$ $(I=2)$
$m_\pi\, a_{\pi\pi}^{I=2}$
m_π / f_π
0
-0.1
-0.2
-0.3
-0.4
-0.5
1
2
3
4
χ - PT (Tree Level)
ETM (L=2.1 fm, a=0.086 fm)
ETM (L=2.7 fm, a=0.086 fm)
ETM (L=2.1 fm, a=0.067 fm)
NPLQCD (MILC fine)
NPLQCD (MILC coarse)
CP-PACS (2004) ($n_f = 2$)
E 865 (2003)

Chiral *and* continuum extrapolation

$$m_\pi a_{\pi\pi}^{I=2}(b \neq 0) \;=\; -\frac{m_\pi^2}{8\pi f_\pi^2}\left[\,1\;+\;\frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(\,3\log\frac{m_\pi^2}{16\pi^2 f_\pi^2}\;-\;1\;-\;l_{\pi\pi}^{I=2}\,\right)\right]$$

Chiral *and* continuum extrapolation

$$m_\pi a_{\pi\pi}^{I=2}(b \neq 0) = -\frac{m_\pi^2}{8\pi f_\pi^2}\left[1 + \frac{m_\pi^2}{16\pi^2 f_\pi^2}\left(3\log\frac{m_\pi^2}{16\pi^2 f_\pi^2} - 1 - l_{\pi\pi}^{I=2}\right)\right]$$



MAχ–PT

$$+ \frac{m_\pi^2}{8\pi f_\pi^2}\left[\frac{1}{(4\pi f_\pi)^2}\left[\frac{\tilde{\Delta}_{ju}^4}{6m_\pi^2}\right]\right]$$

$$\tilde{\Delta}_{ju}^2 \equiv \tilde{m}_{jj}^2 - m_{uu}^2 = 2B_0(m_j - m_u) + b^2\Delta_I + \ldots = 0.0769(22)$$

- Contains all $\mathcal{O}(m_\pi^2 b^2)$ and $\mathcal{O}(b^4)$ lattice artifacts.
- m_π and f_π are the lattice-physical parameters.
- Many sources of systematic error.

Error budget

- Higher-order effects in MAχPT:

$$\mathcal{O}(m_\pi^4 b^2) \sim \frac{2\pi m_\pi^4}{(4\pi f_\pi)^4} \frac{b^2 \Delta_{\rm I}}{(4\pi f_\pi)^2} < 1\%$$

- Finite-volume effects: $\sim 4\%$ at lightest mass.

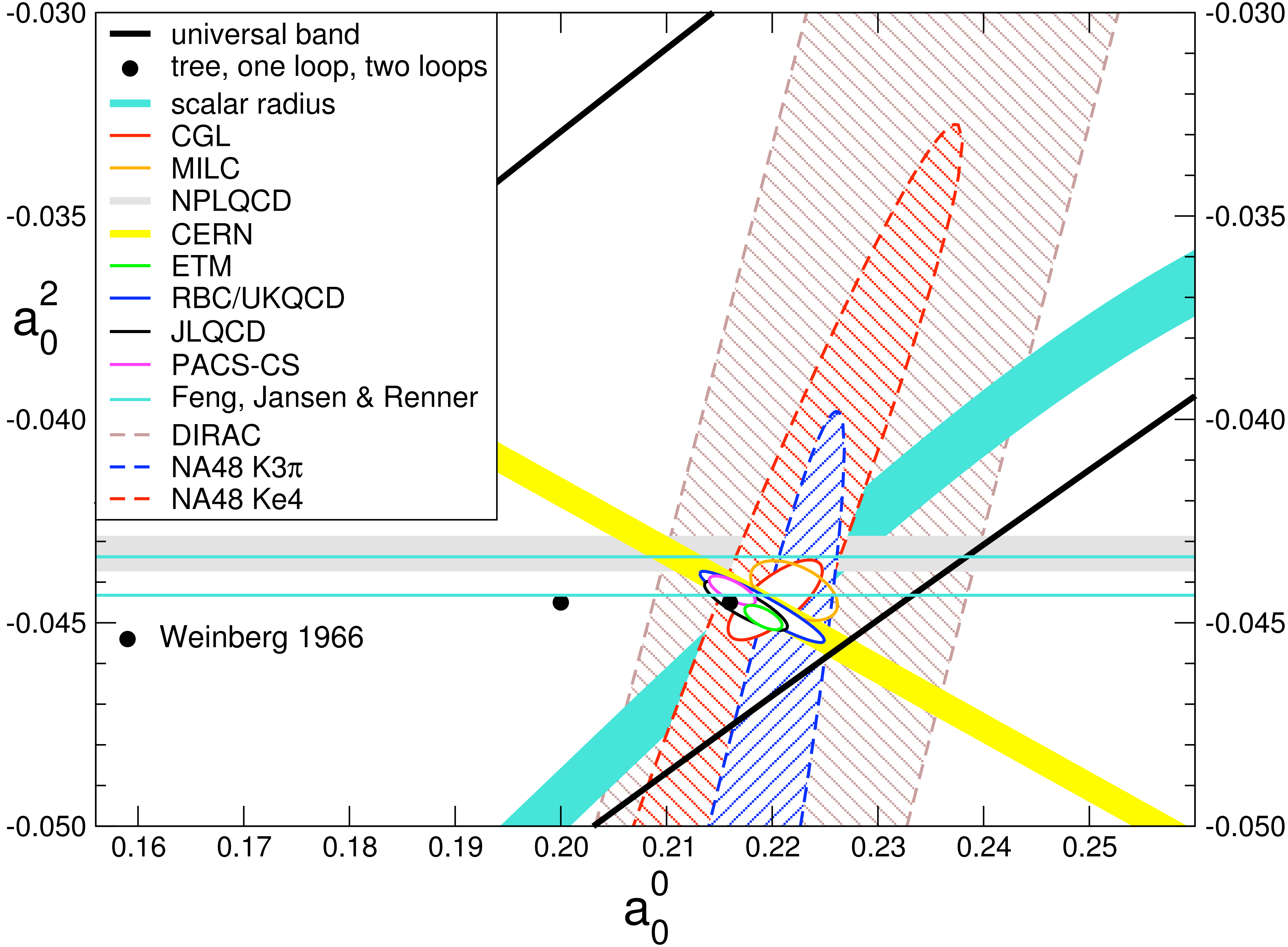
- Residual chiral symmetry breaking:

$$\frac{8\pi m_\pi^4}{(4\pi f_\pi)^4} \frac{m_{res}}{m_l} \sim 3\%$$

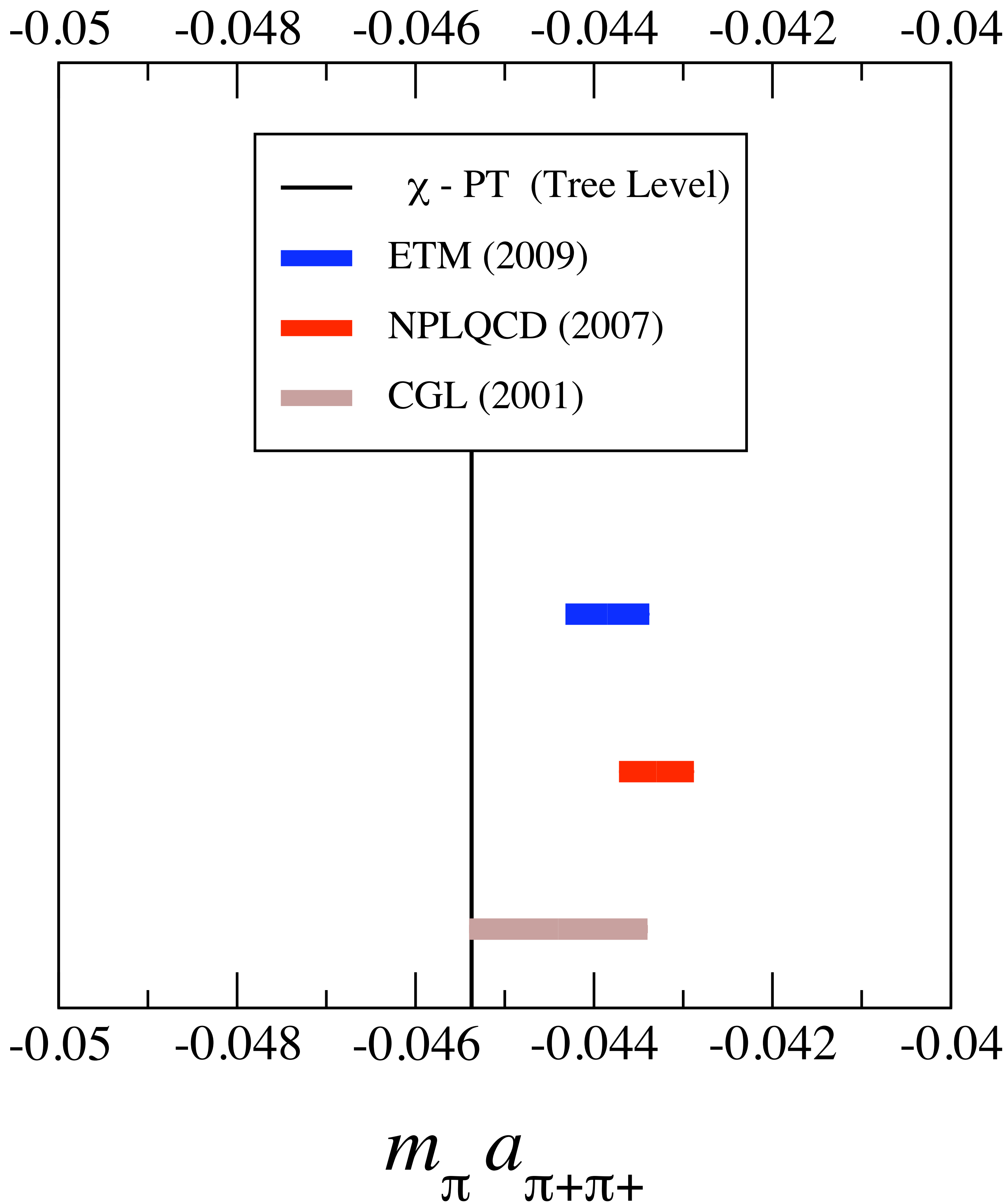
- Range corrections:

$$\frac{(m_\pi a_{\pi\pi}^{I=2})^2 p^2}{2 m_\pi^2} \sim 1\%$$

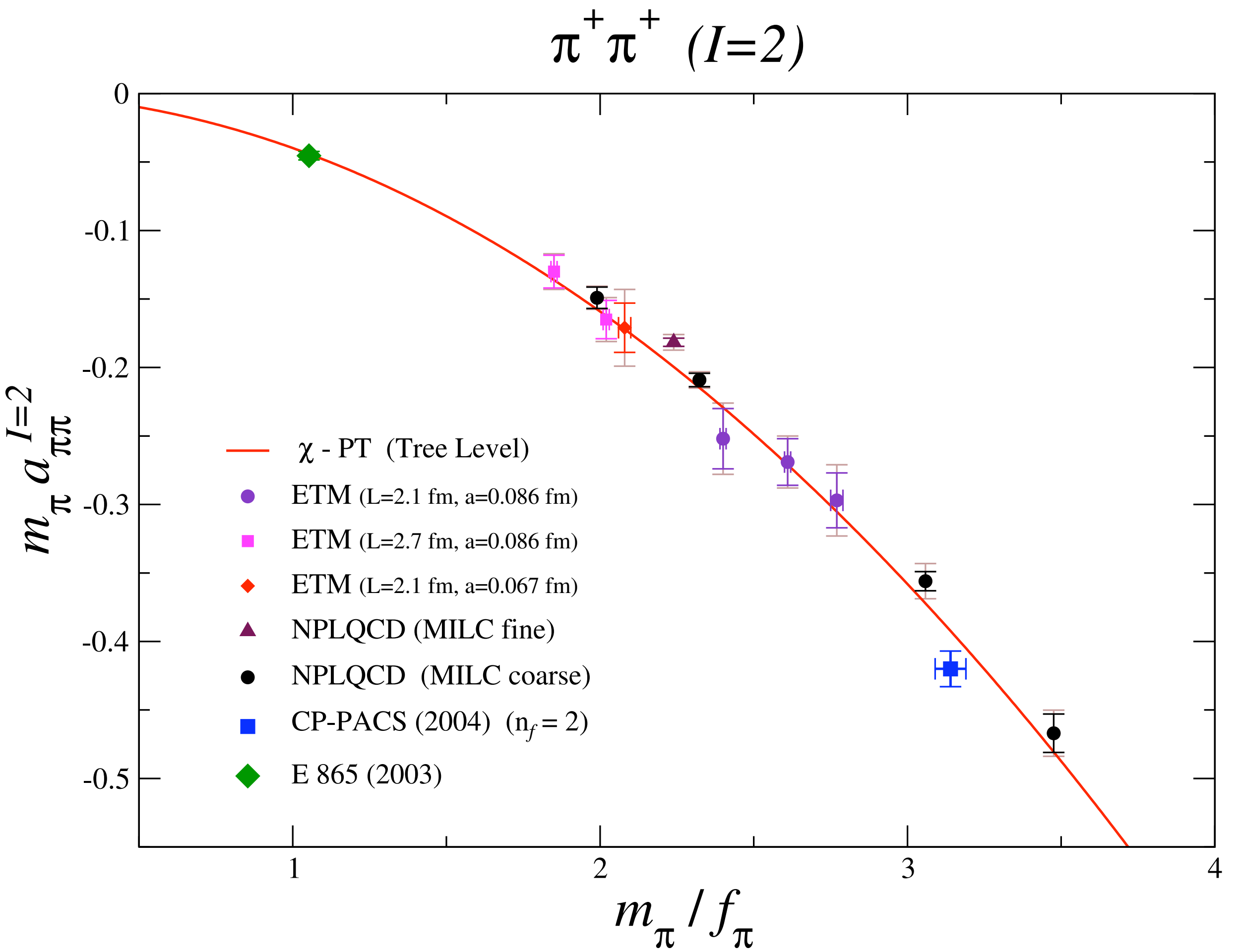
- Isospin violation: Only issue if compare to experiment!



(Courtesy of H. Leutwyler)



-0.05
-0.048
-0.046
-0.044
-0.042
-0.04
χ - PT (Tree Level)
ETM (2009)
NPLQCD (2007)
CGL (2001)
$m_\pi a_{\pi+\pi+}$



EFT works too well!!