# Introduction to Lattice Field Theory

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What these lectures are about....

Fundamentals of lattice field theory: summary of what you can find in several introductory books

"Introduction to Quantum Fields on a Lattice", J. Smit; "Lattice Gauge Theories: An Introduction", H.J. Rothe; "Quantum Fields on a Lattice", Montvay and Münster; "Quarks, gluons and lattices", M. Creutz

- Lecture 1: Functional formulation of Euclidean QFT, regularization, Wilson RG.
- Lecture 2: Scalar and fermion lattice field theories
- Lecture 3: Gauge Field theories on the lattice
- Lecture 4: QCD on the lattice

Selection of recent results → H. Wittig's lectures

Modern perspectives: Proceedings of the Les Houches Summer School "Modern perspectives in lattice QCD"

Elementary particle dynamics are accurately described by Quantum Field Theory (QFT).

LEP $\oplus$ flavour factories have established the Standard Model at $1\%$ or better: SM is a *renormalizable* QFT

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{SSB}$$

$$\begin{aligned}
\mathcal{L}_{gauge} &= -\frac{1}{4g^2_{U(1)}} B_{\mu\nu} B_{\mu\nu} - \frac{1}{4g^2_{SU(2)}} W_{\mu\nu} W_{\mu\nu} - \frac{1}{4g^2_{SU(3)}} G_{\mu\nu} G_{\mu\nu} \\
\mathcal{L}_{matter} &= \sum_a \bar{\Psi}^a i \not{D} \Psi^a \\
\mathcal{L}_{SSB} &= \sum_{ab} \bar{\Psi}^a Y_{ab} \Phi \Psi^b + h.c. + \mathcal{L}(\Phi)
\end{aligned}$$

Most of the *ugly/intriguing* features of the SM are related to the SSB flavour sector that will be soon tested at the LHC:

| Sector | Free Param. | Discrete Sym. | Flavour Sym. |
|---|---|---|---|
| Gauge | 3 | $C, P, T$ | |
| Gauge+matter | 3 | $T, \not{C}, \not{P}$ | $U(N_{\rm f})$ |
| Gauge+matter+SSB | 22-24 | $\not{C}, \not{P}, \not{T}$ | $U(1)_{\rm B-L}$ or none |

Most of what we can predict accurately in this model has been obtained in perturbation theory (PT)

PT is not enough to understand the SM and to go beyond

- Processes involving $SU(3)$ interactions

$$\mathcal{L}_{QCD} = -\frac{1}{4g^2_{SU(3)}} G_{\mu\nu} G_{\mu\nu} + \sum_i \bar{q}_i (i\not{D} + m_i) q_i$$

Responsible for confinement, mass gap, spontaneous chiral symmetry breaking,... and key to phenomenology in quark flavour sector

- SM in extreme conditions of density and temperature (early Universe)

- $\mathcal{L}(\Phi)$ completely untested. *Triviality problem*: $V(\Phi) = -\frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$

$$\lim_{\Lambda \to \infty} \lambda_R = 0$$

SM can only be an effective theory!

- What if the SM is not the whole story ?

SUSY $\leftrightarrow$ non-perturbative effects to break SUSY ?
Technicolor $\leftrightarrow$ up-scaled versions of QCD
Nearly conformal FT ....

- Hint for the origin of chirality and $\not{P}$ of the weak interactions ?

The only first-principles method to define a QFT non-perturbatively is the regularization on a space-time grid

- provides a non-perturbative definition of QFT (at least for those of QCD type)

- can be treated by numerical methods

YM and QCD are benchmarks before exploring other possibilities

# Lecture I: Functional formulation of Euclidean QFT, Regularization and Wilson RG

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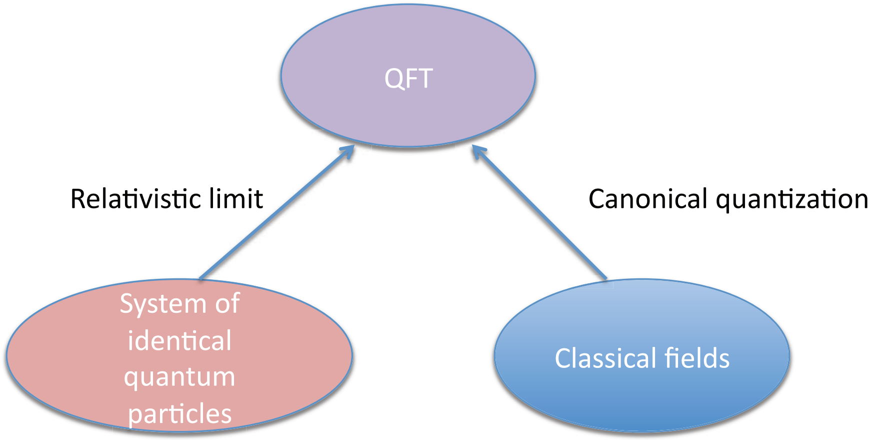



## Fock space

States: $|0\rangle$ $\oplus$ creation/anhilation operators $\hat{a}_i, \hat{a}_i^\dagger$ $\oplus$ conmutation relations $[\hat{a}_i, \hat{a}_j] = 0, [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ ( for bosons )

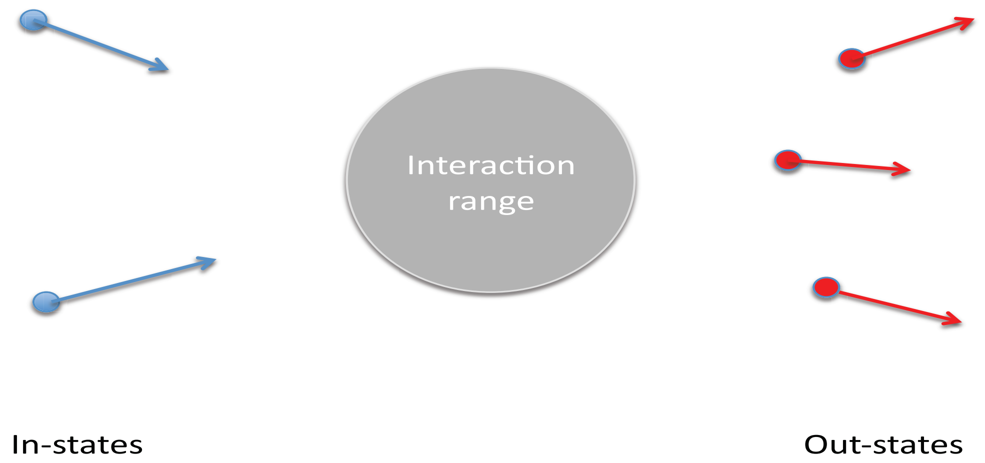
1) Operators: cluster decomposition, causality, hermiticity $\rightarrow$ functions of field operators

2) Canonical quantization of fields $\rightarrow$ bunch of harmonic oscillators: ladder operators $\rightarrow$ creation/anhilation operators in Fock space

The Theory of Quantum Fields I, S. Weinberg

# Lehman-Symanzik-Zimmerman Reduction Formula

Physical observables (cross sections. decav widths) $\leftrightarrow$ Field correlation functions




$$\prod_{i=1}^{n}\int d^4x_i e^{ip_i\cdot x_i}\prod_{j=1}^{k}\int d^4y_j e^{-iq_j\cdot y_i}\langle 0|T\left(\hat{\phi}(x_1)\ldots\hat{\phi}(x_n)\hat{\phi}(y_1)\ldots.\hat{\phi}(y_k)\right)|0\rangle$$

$$\simeq_{p_i^0\to E_{\mathbf{p}_i},q_j^0\to E_{\mathbf{q}_j}}\prod_{i=1}^{n}\left(\frac{i\sqrt{Z}}{p_i^2-m^2+i\epsilon}\right)\prod_{j=1}^{k}\left(\frac{i\sqrt{Z}}{q_j^2-m^2+i\epsilon}\right)\langle\mathbf{p}_1,\ldots.,\mathbf{p}_n,out|\mathbf{q}_1,\ldots,\mathbf{q}_k;in\rangle,$$

$Z$, $m$ one-particle field renormalization constant and mass ?

## Källen-Lehmann Representation of the Propagator

In full generality

$$\langle 0|T\hat{\phi}(x)\hat{\phi}(0)|0\rangle \quad = \quad \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)}\, e^{-ip\cdot x}\Big|_{p_0=E_{\mathbf{p}}(\alpha)} |\langle 0|\hat{\phi}(0)|\alpha(\mathbf{p})\rangle|^2$$

$$E^2_{\mathbf{p}}(\alpha) = m(\alpha)^2 + \mathbf{p}^2$$

$\sum_\alpha \leftrightarrow$ sum over one-particle states (discrete) and multiparticle states (continuum):

$$Z_\alpha \equiv |\langle 0|\hat{\phi}(0)|\alpha(\mathbf{0})\rangle|^2 \quad m(\alpha) = \text{Energy zero momentum}$$

The $Z_\alpha$ and $m(\alpha)$ of the one-particle states characterize the asymptotic states in the LSZ formula.

## Wick rotation

Time-ordered correlation functions contain all the physical information of the theory

$$W_n(t_1, \mathbf{x_1}; ...., t_n, \mathbf{x}_n) \quad = \quad \langle 0|\hat{\phi}(t_1, \mathbf{x}_1)...\hat{\phi}(t_n, \mathbf{x}_n)|0\rangle, \quad t_1 \geq t_2..... \geq t_n,$$

Can be continuously extended to analytic functions in the complex plane for

$$\mathrm{Im} t_1 \leq \mathrm{Im}\ t_2 \leq .... \leq \mathrm{Im}\ t_n$$

The Schwinger functions or Euclidean correlation functions are defined as:

$$S_n(x_1, ..., x_n) = W_n(-ix_1^0, \mathbf{x}_1; ... - ix_n^0, \mathbf{x}_n),$$

where the Euclidean times are $x_i^0 = it_i^0$ and

$$x_1^0 \geq x_2^0.... \geq x_0^n.$$

## Functional Formulation of quantum mechanics $\to$ in terms of classical variables

The quantum operator that evolves states from time $t_i$ to $t_f$ is

$$\hat{U}(t_f, t_i) = e^{-i\hat{H}(t_f - t_i)}$$

Consider $\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$

Let us divide the time interval in a large number, $N$, of infinitesimal intervals of width $\tau$:

$$t_n = t_i + n\tau, \quad n = 0, ..., N, \quad \tau \equiv \frac{t_f - t_i}{N}.$$

$$\hat{U}(t_f, t_i) = \hat{U}(t_f, t_{N-1})\hat{U}(t_{N-1}, t_{N-2})....\hat{U}(t_1, t_i) = \hat{T}^N \quad \hat{U}(t_{n+1}, t_n) = e^{-i\hat{H}\tau} \equiv \hat{T}.$$

At each time slice $t_n$ the identity operator can be chosen as

$$\hat{1} = \int d^3x_n \; |\mathbf{x}_n\rangle\langle\mathbf{x}_n|,$$

$$\hat{U}(t_f, t_i) \quad = \quad \left(\prod_{n=1}^{N-1} \int d^3x_n\right) \hat{T}|\mathbf{x}_{N-1}\rangle \left(\prod_{n=2}^{N-1} \langle\mathbf{x_n}|\hat{T}|\mathbf{x_{n-1}}\rangle\right) \langle\mathbf{x_1}|\hat{T}$$

*Define* a new transfer operator $\hat{T}_F$ that coincides with $\hat{T}$ in the limit $\tau \to 0$:

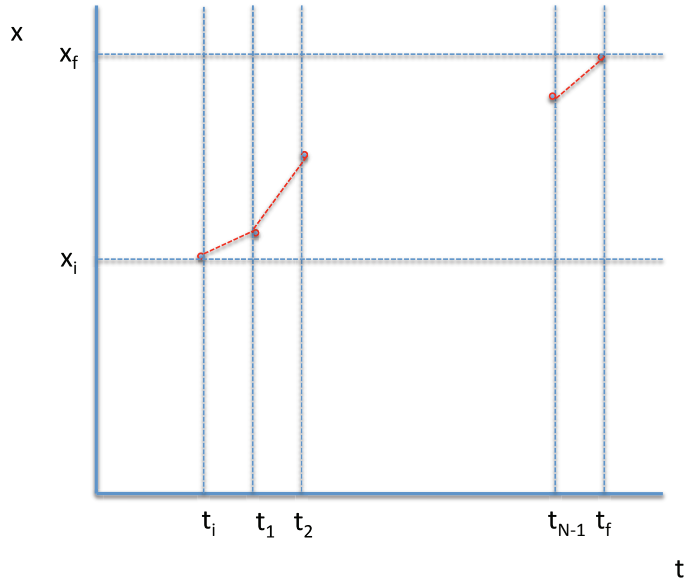
$$\hat{T}_F \equiv e^{-i\frac{\tau}{2}V(\hat{x})} \; e^{-i\tau\frac{\hat{P}^2}{2m}} e^{-i\frac{\tau}{2}V(\hat{x})},$$

$$
\begin{aligned}
\langle \mathbf{x}_{n+1} | \hat{T}_F | \mathbf{x}_n \rangle &= \sqrt{\frac{m}{2\pi i \tau}} \exp \left[ i\tau \left( \frac{m}{2} \left( \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\tau} \right)^2 - \frac{V(\mathbf{x}_{n+1}) + V(\mathbf{x}_n)}{2} \right) \right] \\
&= \sqrt{\frac{m}{2\pi i \tau}} e^{i\tau \mathcal{L}(t_n)},
\end{aligned}
$$

$\mathcal{L}$ is the time-discretized classical Lagrangian

$$
\mathbf{x}(t_n) = \mathbf{x}_n \quad \mathcal{L}(t) \equiv \frac{1}{2} m \left( \frac{d\mathbf{x}(t)}{dt} \right)^2 - V(\mathbf{x}(t))
$$

Feynman




Amplitude as a integral over paths between $(t_i, \mathbf{x}_i)$ and $(t_f, \mathbf{x}_f)$:

$$
\begin{aligned}
\langle \mathbf{x}_f | \hat{U}(t_f, t_i) | \mathbf{x}_i \rangle &= \lim_{N\to\infty} \left( \sqrt{\frac{m}{2\pi i \tau}} \right)^N \prod_{n=1}^{N} \int d^3 x_n e^{i\tau \sum_{n=0}^{N-1} \mathcal{L}(t_n)} \Big|_{\mathbf{x}(t_f)\equiv\mathbf{x}_f ; \mathbf{x}(t_i)\equiv\mathbf{x}_i} \\
&\equiv c \int \mathcal{D}x(t)\; e^{i\int_{t_i}^{t_f} \; dt \mathcal{L}(t)}
\end{aligned}
$$

Important observation: there is no proof of the equivalence between the two representations (canonical and functional), they are *alternative representations*

The way back from functional integrals to quantum operators in Fock space is via the transfer operator $\hat{T}_F$ that must be positive

$$\hat{H}_F \equiv \frac{i}{\tau} \log \hat{T}_F$$

$\hat{H}_F$ and $\hat{H}$ do not coincide, although they are expected to lead to the same physics.

## Functional Formulation of Scalar QFT

| QM | QFT |
|---|---|
| $\{\hat{\mathbf{x}}, \hat{\mathbf{p}}\}$ | $\{\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{x})\}$ |
| $\hat{1} = \int d^3x \ \|\mathbf{x}\rangle\langle\mathbf{x}\|$ | $\hat{1} = \int \prod_{\mathbf{x}} d\phi(\mathbf{x}) \ \|\phi\rangle\langle\phi\|$ |

$$\hat{U}(t_f, t_i) \quad = \quad \hat{T}^N, \quad N\tau = t_f - t_i.$$

$$\hat{U}(t_f, t_i) = \int \prod_{n=1}^{N-1} d\phi_n(\mathbf{x}_n) \ \hat{T}|\phi_{N-1}\rangle\langle\phi_{N-1}|\hat{T}...|\phi_1\rangle\langle\phi_1|\hat{T}$$

$$\hat{T} \equiv \exp\left(-\frac{\tau}{2}\hat{H}_V\right)\exp\left(-\tau\hat{H}_K\right)\exp\left(-\frac{\tau}{2}\hat{H}_V\right),$$

$$\hat{H}_V \equiv \int d^3x \, \left[\frac{1}{2}(\nabla\hat{\phi})^2 + V(\hat{\phi})\right]. \quad \hat{H}_K \equiv \int d^3x \, \frac{1}{2}\hat{\pi}^2$$

$$\langle\phi_{n+1}|\hat{T}|\phi_n\rangle \quad = \quad \exp\left(-\tau\mathcal{L}(\phi_n)\right)$$

$$\begin{aligned}\langle\phi_f|\hat{U}(t_f,t_i)|\phi_i\rangle \quad &= \quad \lim_{N\to\infty}\int\left[\prod_{n=0}^{N}\prod_{\mathbf{x}_n} d\phi_n(\mathbf{x_n})\right]\exp\left(-\tau\sum_{n=0}^{N}\mathcal{L}(\phi_n))\right) \\ &\equiv \quad \int_{\substack{\phi(\mathbf{x},t_i)=\phi_i(\mathbf{x}) \\ \phi(\mathbf{x},t_n)=\phi_f(\mathbf{x})}} \mathcal{D}\phi\exp\left(-\int dt\mathcal{L}(\phi)\right)\end{aligned}$$

The partition function

$$\mathcal{Z} \equiv \mathrm{Tr}[\hat{U}(T/2,-T/2)] = \lim_{N\to\infty} \mathrm{Tr}\left[\hat{T}^N\right] = \int_{PBC} \mathcal{D}\phi \; e^{-\mathcal{S}[\phi]},$$

$$S[\phi] = \int dt \mathcal{L}(\phi) = \int d^4x \left\{ \frac{1}{2} \left(\partial_\mu \phi(x)\right)^2 + V(\phi(x)) \right\}$$

Correlation functions in functional formalism (if $|0\rangle$ lowest energy state):

$$\langle 0|\hat{O}|0\rangle = \lim_{T\to\infty} \frac{\mathrm{Tr}\left[\hat{O}e^{-\hat{H}T}\right]}{\mathrm{Tr}\left[e^{-\hat{H}T}\right]} = \lim_{T\to\infty} \frac{\mathrm{Tr}\left[\hat{O}e^{-\hat{H}T}\right]}{\mathcal{Z}},$$

$$S_n = \langle 0|\hat{\phi}(\mathbf{x}_1, t_1)....\hat{\phi}(\mathbf{x}_n, t_n)|0\rangle = \lim_{T\to\infty} \mathrm{Tr}\left[\hat{\phi}(\mathbf{x}_1, t_1)...\hat{\phi}(\mathbf{x}_n, t_n)e^{-\hat{H}T}\right]/\mathcal{Z}.$$

The same procedure of discretizing time and

$$S_n = \frac{\int_{PBC} \mathcal{D}\phi\ e^{-S[\phi]}\phi(\mathbf{x}_1, t_1)....\phi(\mathbf{x}_n, t_n)}{\int_{PBC} \mathcal{D}\phi\ e^{-S[\phi]}} \equiv \langle \phi(x_1)....\phi(x_n)\rangle,$$

where the integrals are over periodic classical fields, as defined above.

## Generating functional of correlation functions

$$Z[J] = \langle e^{\int d^4x J(x)\phi(x)} \rangle,$$

$$\frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0} = \langle \phi(x_1)\dots\phi(x_n) \rangle \quad \frac{\delta}{\delta J(x)} J(y) = \delta(x-y).$$

Free case ($V(\phi) = \frac{1}{2} m_0^2 \phi^2$)

$$Z[J] \sim \exp\left( \frac{1}{2} \int d^4x d^4y J(x) K^{-1}(x,y) J(y) \right), \qquad K \equiv -\partial_\mu^2 + m_0^2,$$

The free propagator is

$$\langle \phi(x)\phi(y) \rangle = \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)}\Big|_{J=0} = K^{-1}(x,y) = \int d^4p \, \frac{e^{i(x-y)}}{p^2 + m_0^2}.$$

In the interacting case no exact solution:

- Perturbation theory

- Non-perturbative evaluation of the correlation functions via a discretization of space-time: lattice formulation.

$$\text{QFT} \leftrightarrow \text{Statistical system}$$

## Perturbation Theory in the Functional Formalism

$$V(\phi) = \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4 \qquad S[\phi] = S^{(0)}[\phi] + S^{(1)}[\phi]$$

$$S^{(0)}[\phi] \equiv \int d^4x \frac{1}{2}\left[(\partial_\mu\phi(x))^2 + m_0^2\phi^2\right], S^{(1)}[\phi] = \int d^4x \frac{\lambda}{4!}\phi^4$$

The n-th Schwinger function is given by

$$S_n = \frac{\langle \phi(x_1)\phi(x_2)...\phi(x_n)e^{-S^{(1)}[\phi]}\rangle_0}{\langle e^{-S^{(1)}[\phi]}\rangle_0}, \qquad e^{-S^{(1)}[\phi]} = \sum_n \frac{1}{n!}\left(-S^{(1)}[\phi]\right)^n$$

For each insertion of $S^{(1)}[\phi]$:

$$\int d^4x \leftrightarrow \int d^4p \leftrightarrow \mathrm{UV\ divergences}$$

## Perturbative renormalizability

The contribution of a 1PI diagram with $I$ internal lines (i.e. propagators linking two vertices) and $L$ loops is generically of the form:

$$\Gamma^{(N)}(p_1, ..., p_N) \sim \int \prod_{l=1}^{L} d^4 q_l \prod_{i=1}^{I} \frac{1}{k_i(q_l, p_j)^2 + m^2}.$$

*Superficial degree of divergence*: if $q_i \sim \Lambda \to \Gamma^{(N)} \sim \Lambda^\omega$

$$\omega \equiv 4L - 2I,$$

Negative $\omega$ is necessary for UV finiteness but not sufficient!

Topological relation between $I$, the number of vertices $V$ and external legs $N$ of the diagram:

$$2I + N = 4V,$$

Finally the number of loops, $L$, is related to $V$ and $N$:

$$L = I - V + 1 = V - N/2 + 1 \rightarrow \quad \omega = 4 - N$$

1PI diagrams with $N = 2, 4$ might have a positive $\omega$. It can be shown that the UV divergences in these diagrams can be reabsorbed in a redefinition of $m_0^2$, $\lambda$ and the normalization of the field itself. For this reason, we say that this theory is *perturbatively renormalizable*

More generically, we can consider a theory where $S^{(1)}$ has other interactions such as

$$V^{(1)}[\phi] = g_V \, (\partial)^{N_\partial} \, (\phi)^{N_\phi}$$

$$\omega = 4 - N - [g_V]V, \quad [g_V] = 4 - N_\phi - N_\partial$$

A very different behaviour as the order of the perturbative expansion grows depending on the sign of $[g_V]$:

| | |
|---|---|
| $[g_V] > 0$ | diagrams become less divergent with $V$: *superrenormalizable theory* |
| $[g_V] = 0$ | the divergence does not depend on $V$: *renormalizable theory* |
| $[g_V] < 0$ | divergences for larger $N$ as $V$ grows: *non-renormalizable theory* |

The lattice formulation of any lattice field theory is not renormalizable in this sense....

## Wilsonian renormalization group

There is nothing special in a bare Lagrangian that is renormalizable ...

Renormalizability is an emergent effective phenomenon: if a theory describes correlation lengths that tend to infinity in units of the cutoff, it can be accurately represented by a renormalizable theory (as long as we are interested in describing physics at scales of the order of this long correlation length)

Renormalization group transformations

K. Wilson studied the connection of renormalizability and critical phenomena via his celebrated *renormalization group transformations*

Fundamental cutoff: $a = \Lambda^{-1}$ the existence of a continuum limit for any physical scale $m_{phys} = \xi^{-1}$ implies

$$m_{phys} a \to 0 \qquad \leftrightarrow \qquad \xi/a \to \infty$$

Continuum limit of a QFT $\leftrightarrow$ Critical point of the statistical system

Empirical fact: many systems near critical points behave in similar ways, this is what is called *universality* (the long range properties of many systems do not depend on the details of the microscopic interactions)

Renormalizability in QFT $\leftrightarrow$ Universality in critical statistical systems

Both phenomena can be understood in terms of *fixed-points* of the renormalization group.

## Renormalization group transformations

Let us suppose that we have a lattice scalar theory on a lattice of spacing $a$ which describes physics scales $m \ll a^{-1}$.

The most general local theory:

$$S(a) = \sum_{\alpha} g_{\alpha}(a) \sum_{x} O_{\alpha}(\phi(x), a)$$

where $O_{\alpha}$ are Lorentz invariant and local operators with arbitrary dimension constructed by powers of $\partial_{\mu}\phi$, $\phi$ and $a$.

Take the limit $a \to 0$ in little steps:

$$a \geq a_1 \geq a_2 ... \geq a_n = (1-\epsilon)^n a, \quad \epsilon \ll 1$$

At each step we can integrate the modes between $a_{n-1}^{-1}$ and $a_n^{-1}$ to obtain an effective theory at a lower scale:

$$
\begin{aligned}
S(a_1) &\rightarrow S^{(1)}(a) = \sum_\alpha g_\alpha^{(1)}(a) \sum_x O_\alpha(\phi(x), a) \\
S(a_2) &\rightarrow S^{(1)}(a_1) \rightarrow S^{(2)}(a) = \sum_\alpha g_\alpha^{(2)}(a) \sum_x O_\alpha(\phi(x), a) \\
.... & \\
S(a_n) &\rightarrow .... \sum_\alpha g_\alpha^{(n)}(a) \sum_x O_\alpha(\phi(x), a)
\end{aligned}
$$

The operators at scale $a$ are all the same because we included *all* possible

*Renormalization group (RG) transformation*, the function that defines the change in the couplings:

$$
R_\alpha : g_\alpha^{(n)} \rightarrow g_\alpha^{(n+1)} \quad g_\alpha^{(n+1)} = R_\alpha(g^{(n)})
$$

For a continuous transformation there is a RG flow of the coupling constants

*Fixed-point* corresponds to some point in coupling space $g_\alpha^*$:

$$R_\alpha(g^*) = g_\alpha^*.$$

Physics no longer changes as we move towards the continuum limit, since the action remains unchanged.

Fixed-points are therefore critical points:

$$\lim_{n\to\infty} m_{phys}(g^*)a_n \to 0$$

Fixed-points, if they exist, are rather universal: the approach to such points can be achieved by tuning just a few parameters that are called relevant or marginal.

Near a fixed-point the evolution of the couplings reads at linear order

$$g_\alpha^{(n+1)} - g_\alpha^* = \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*} (g_\beta^{(n)} - g_\beta^*),$$

so the distance to the fixed-point $\Delta g^{(n)}$ changes according to the following equation:

$$\Delta g_\alpha^{(n+1)} \quad = \quad M_{\alpha\beta} \Delta g_\beta^{(n)}, \qquad M_{\alpha\beta} \equiv \left.\frac{\partial R_\alpha}{\partial g_\beta}\right|_{g^*}.$$

We can find different situations depending on the eigenvalues, $\lambda$, of the matrix $M$:

| | | |
|---|---|---|
| $\lambda > 1$ | $\Delta g_\alpha^{(n)}$ increases as $n \to \infty$ | $\alpha$ is a relevant direction |
| $\lambda = 1$ | $\Delta g_\alpha^{(n)}$ stays the same as $n \to \infty$ | $\alpha$ is a marginal direction |
| $\lambda < 1$ | $\Delta g_\alpha^{(n)}$ decreases as $n \to \infty$ | $\alpha$ is an irrelevant direction |

The fact that the number of relevant directions is finite and usually small is behind the two related properties: universality of the fixed-point and the renormalizability of the corresponding QFT.

## Example: Gaussian Fixed Point

Case 1: the free massless point of a scalar theory is a fixed-point:

$$S(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi(-p) p^2 \phi(p),$$

where $BZ(a)$ is the Brillouin zone $[-\pi/a, \pi/a]$ in each mometum direction.

When we do the first RG transformation we start with the same action but in a lattice of spacing $a_1 = (1-\epsilon)a$. Since the fields at different momenta are independent variables, we can integrate over those at momenta $\pi/a \leq |p_\mu| \leq \pi/a_1$ so that the partition function:

$$\mathcal{Z}^{(1)} = \int \prod_{p \in BZ(a_1)} d\phi(p) e^{-\int_{BZ(a_1)} \frac{d^4p}{(2\pi)^4} \frac{1}{2}\phi(-p)p^2\phi(p)}$$
$$= C \int \prod_{p \in BZ(a)} d\phi(p) e^{-\int_{BZ(a)} \frac{d^4p}{(2\pi)^4} \frac{1}{2}\phi(-p)p^2\phi(p)}.$$

$C$ is some constant that comes from the integration of the momentum modes of $BZ(a_1)$ that lay out of $BZ(a)$

The effective action after integrating the high frequency modes is therefore $S^{(1)}(a) = S(a)$. The original action is a fixed-point of the RG.

There is no mass term $\rightarrow$ massless theory (critical point)

Case 2: We start with an arbitrary lattice action that is quadratic in the fields, but including all terms that are Lorentz invariant.

$$S(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi(-p) \left( p^2 + m_0^2 \frac{1}{a^2} + g_1 a^2 p^4 + ... \right) \phi(p)$$

$$[m_0] = [\alpha] = ... = 0.$$

the integration over the momentum modes in a slice of momenta in $BZ(a_1)$ and out of $BZ(a)$ can be done as before

$$S^{(1)}(a) = \int_{BZ(a)} \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi(-p) \left( p^2 + \left(\frac{a}{a_1}\right)^2 \frac{1}{a^2} m_0^2 + g_1 \left(\frac{a_1}{a}\right)^2 a^2 p^4 + ... \right) \phi(p),$$

the action is no longer a fixed-point:

$$\begin{pmatrix} {m_0^{(1)}}^2 \\ g_1^{(1)} \\ \ldots \end{pmatrix} = M \begin{pmatrix} m_0^2 \\ g_1 \\ \ldots \end{pmatrix}, \quad M = \begin{pmatrix} (1-\epsilon)^{-2} & 0 & \ldots \\ 0 & (1-\epsilon)^2 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix},$$

Only one eigenvalue of $M$ is above one : only one relevant direction, $m_0^2$

After a large number of RG transformations (as we approach the continuum limit) all directions disappear, except $m_0^2$ which fixes the physical mass gap and needs to be tuned to remain finite in the continuum limit.

The continuum limit of this theory: a free massive renormalizable scalar QFT

Case 3: In this case the action contains all terms, including interactions

$$S(a) = \sum_x \partial_\mu \phi \partial_\mu \phi + \frac{1}{2a^2} m_0^2 \phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\lambda'}{6!}\phi^6 + g_1 a^2 \phi \partial^4 \phi + ....$$

The integration over the momentum shell $\pi/a \leq |p_\mu| \leq \pi/a_1$ cannot be done analytically. For sufficiently small couplings it can be done in perturbation theory:

$$\begin{aligned}
Z^{(1)} &= 1 + \mathcal{O}(\lambda^2), \\
{m_0^{(1)}}^2 &= (m_0^2 + \delta m_0^2)(1-\epsilon)^{-2} \\
\lambda^{(1)} &= \lambda + \delta\lambda \\
\lambda'^{(1)} &= (\lambda' + \delta\lambda')(1-\epsilon)^2 \\
g_1^{(1)} &= (g_1 + \delta g_1)(1-\epsilon)^2
\end{aligned}$$

where all $\delta$ terms depend on the couplings $\lambda, \lambda', ...$, but vanish for small enough couplings

Eigenvalues of $M$: one relevant $m_0$, one marginal $\lambda$ and the rest irrelevant. $\delta\lambda$, even if small, is important since it determines the fate of this direction:

$$\delta\lambda = \frac{3\lambda^2}{16\pi^2}\log(1-\epsilon) < 0,$$

$\lambda^{(1)} < \lambda$ and the direction is marginally irrelevant.

Continuum theory: a massive free scalar theory (within this perturbative analysis)

Summarizing Wilson's approach

QFT with a cutoff $\leftrightarrow$ Statistical system near criticality
Renormalized QFT $\leftrightarrow$ Statistical system at a fixed-point

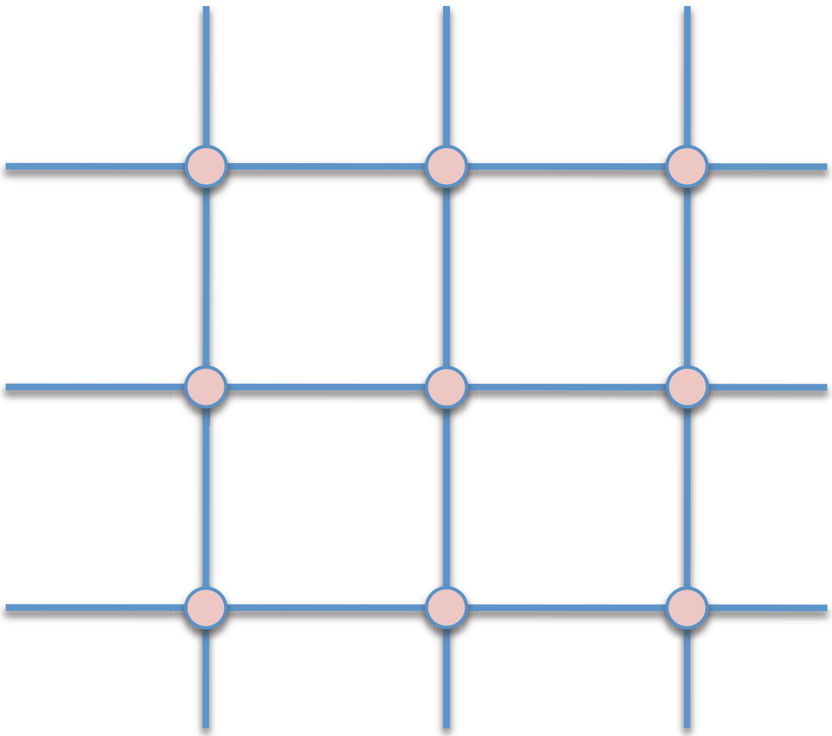
Essential for the definition of QFT on a lattice: for any $S(a)$, the continuum limit will approach the fixed-point of the statistical system nevertheless.

We need to make sure that the FP corresponds to the QFT we want to describe:

- the action has the right degrees of freedom
- it is local
- has the right symmetries to flow to the desired fixed-point (for example if we break some symmetry we might artificially increase the number of relevant directions)

Under these very general assumptions we are otherwise free to make our choice.

# Lecture II: Free quantum matter fields on the lattice



## Lattice Scalar QFT

A scalar field in a discretized space-time, such as a cubic lattice:

$$\phi(x) \qquad x = na \quad n = (n_0, n_1, n_2, n_3) \quad n_i \in Z^4.$$

$$\int dx_i \to a \sum_{n_i \in Z} \qquad \int d^4x \to a^4 \sum_x \equiv a^4 \sum_{n \in Z^4} .$$

Any $F(na)$ has a Fourier series periodic in the Brillouin zone (BZ):

$$\tilde{F}(p) = a^4 \sum_n e^{-ipna} F(na) \quad \tilde{F}(p) = \tilde{F}\left(p + \frac{2\pi}{a} m\right), \;\; m \in Z^4$$

and

$$\int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} e^{ipna} \tilde{F}(p) = F(na).$$

Lattice momenta are cutoff at scale $|p_i| \le \pi/a$ i.e. the theory is regularized.

A very useful formula is Poisson's summation formula:

$$\sum_{n \in Z^4} e^{inz} = (2\pi)^4 \sum_{n \in Z^4} \delta(z - 2\pi n) \equiv (2\pi)^4 \delta_P(z).$$

Momentum is conserved modulo $2\pi/a$.

The functional approach to quantization in Euclidean

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{-S[\phi]}, \quad \mathcal{D}\phi \to \prod_x d\phi(x),$$

$S[\phi]$ is a discretized version of the action $\lambda\phi^4$ continuum action (all actions should be equivalent in the continuum limit):

$$S[\phi] \quad \to \quad a^4 \sum_x \left\{ \frac{1}{2} \hat{\partial}_\mu \phi(x) \hat{\partial}_\mu \phi(x) + \frac{1}{2} m_0^2 \phi(x)^2 + \frac{\lambda}{4!} \phi(x)^4 \right\},$$

*Forward lattice derivative*

$$\hat{\partial}_\mu \phi(x) \equiv \frac{1}{a} \left( \phi(x + \hat{\mu} a) - \phi(x) \right)$$

*Backward derivative*

$$\hat{\partial}^*_\mu \phi(x) \equiv \frac{1}{a} \left( \phi(x) - \phi(x - \hat{\mu} a) \right)$$

As in the continuum we can obtain the correlation functions from the generating functional

$$Z[J] \equiv \int \prod_x d\phi(x) e^{-S[\phi]+a^4\sum_x J(x)\phi(x)}/\mathcal{Z}.$$

## Free Theory ($\lambda = 0$)

$$S^{(0)}[\phi] \quad = \quad a^4 \sum_x \left\{ \frac{1}{2}\hat{\partial}_\mu \phi \hat{\partial}_\mu \phi + \frac{m_0^2}{2}\phi^2 \right\} = \frac{a^4}{2}\sum_{x,y}\phi(x) K_{xy}\phi(y),$$

$$K_{xy} \equiv -\frac{1}{a^2}\sum_{\hat{\mu}=0}^{3}(\delta_{x+a\hat{\mu}y} + \delta_{x-a\hat{\mu}y} - 2\delta_{xy}) + m_0^2\delta_{xy}$$

$$Z^{(0)}[J] = e^{\frac{a^4}{2}\sum_{x,y} J_x (K^{-1})_{xy} J_y} \det\left(a^4 K\right)^{-1},$$

The propagator: $\langle \phi(x)\phi(y)\rangle_0 = \frac{1}{a^8}\frac{\partial Z^{(0)}[J]}{\partial J_x \partial J_y}\Big|_{J=0} = \frac{1}{a^4} K^{-1}_{xy}$

In Fourier space:

$$\tilde{K}_{pq} \quad = \quad a^8 \sum_{xy} e^{-ipx} e^{-iqy} K_{xy} = a^4 (2\pi)^4 \delta_P(p+q) \left\{ m_0^2 + \sum_\mu \hat{p}_\mu^2 \right\},$$

$$\hat{p}_\mu \equiv \frac{2}{a}\, \sin\left(\frac{p_\mu a}{2}\right) \qquad \hat{p}^2 \equiv \sum_\mu \hat{p}_\mu^2.$$

$$\langle \phi(x)\phi(y)\rangle = a^{-4} K^{-1}_{xy} = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{\hat{p}^2 + m_0^2}.$$

## Physical interpretation and unitarity

It is instructive to understand in this very simple context two important questions:

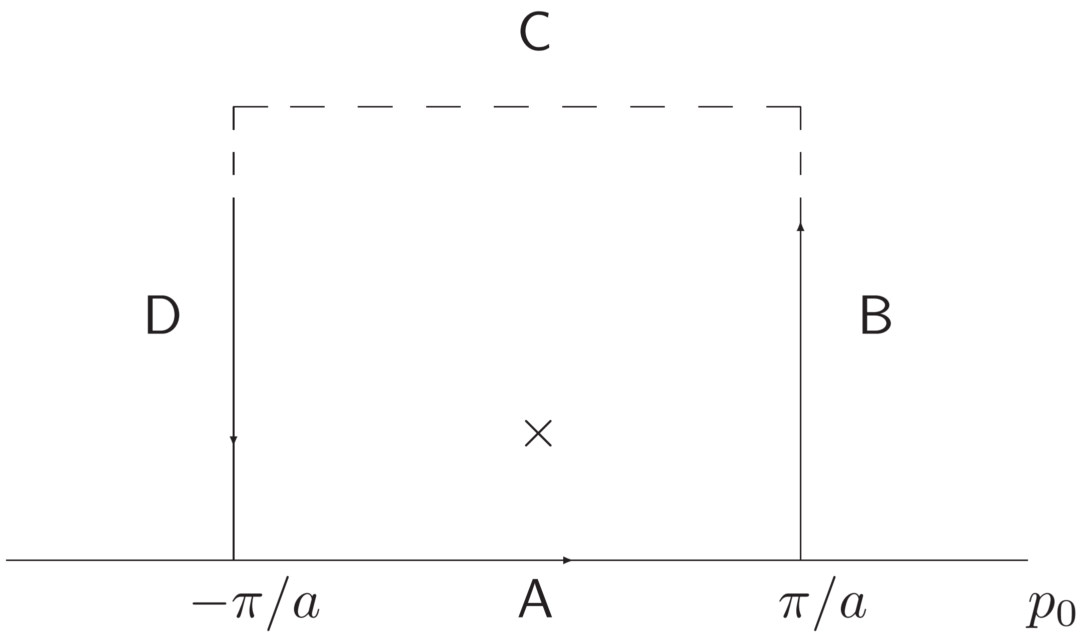
- what is the particle interpretation ?
- what happens in the continuum limit ?

One-particle asymptotic states from the Källen-Lehmann spectral representation

$$\lim_{x_0 \to +\infty} \langle \phi(x)\phi(0) \rangle = \sum_\alpha \int \frac{d^3p}{(2\pi)^3 2E_{\mathbf{p}}(\alpha)} |\langle 0|\hat{\phi}(0)|\alpha(0)\rangle|^2 e^{-E_{\mathbf{p}}(\alpha)x_0} e^{i\mathbf{p}\cdot\mathbf{x}},$$

with $E_{\mathbf{p}}(\alpha) = \sqrt{m_\alpha^2 + \mathbf{p}^2}$.

We can perform the integral over $p_0 \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$ (contour A):




$$\int_A (\ldots) + \int_B (\ldots) + \int_C (\ldots) + \int_D (\ldots) = 2\pi i \sum_{poles} \text{Residues.}$$

By periodicity of the function in the BZ, we have

$$\int_B (...) + \int_D (...) = 0,$$

while for $x_0 > 0$, the integral over $C$ vanishes, $\int_C (...) = 0$:

$$\int_A (...) = 2\pi i \sum_{poles} \text{Residues}.$$

Single poles occur at the solutions of the equation:

$$\hat{p}^2 + m^2 = 0 \Rightarrow p_0 = \pm i\omega(\mathbf{p}) \left( mod \ \frac{2\pi}{a} \right),$$

which are purely complex in the $BZ$. $\omega(\mathbf{p})$ is a real number satisfying:

$$\cosh \omega(\mathbf{p}) a = 1 + \frac{a^2}{2} \left( m_0^2 + \frac{4}{a^2} \sum_{i=1}^{3} \sin^2 \frac{p_i a}{2} \right).$$

There is only one solution within the closed contour with residue

$$\text{Residue}[p_0 = +i\omega(\mathbf{p})] = \frac{1}{2\bar{\omega}(\mathbf{p})}, \qquad \bar{\omega}(\mathbf{p}) \equiv \frac{1}{a}\sinh\left(\omega(\mathbf{p})a\right)$$

$$\langle\phi(x)\phi(0)\rangle = \int_A (\ldots) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\bar{\omega}(\mathbf{p})}e^{-\omega(\mathbf{p})x_0}e^{i\mathbf{p}\cdot\mathbf{x}}.$$

This is the expected behaviour if we identify the one-particle energies $E_{\mathbf{p}}(\alpha) \to \omega(\mathbf{p})$ and

$$|\langle 0|\hat{\phi}(0)|\alpha\rangle| \to \sqrt{\frac{\omega(\mathbf{p})}{\bar{\omega}(\mathbf{p})}}.$$

The continuum limit $a \to 0$ can be readily obtained:

$$\lim_{a\to 0}\ \omega(\mathbf{p}) = \lim_{a\to 0}\ \bar{\omega}(\mathbf{p}) = \sqrt{m_0^2 + \mathbf{p}^2} + \mathcal{O}(a^2)$$

## Interacting theory ($\lambda \neq 0$)

The theory cannot be solved analytically, but the fundamental property of unitarity or the existence and uniqueness of the Hilbert space representation can be proven

- Identifify $\hat{T}$ and $\hat{\phi}$ acting on a a Fock space:

$$\langle \phi(x_1)...\phi(x_n)\rangle = \lim_{T\to\infty} \frac{\mathrm{Tr}\left[\hat{T}^{(T/2-x_1^0)/a}\hat{\phi}(0,\mathbf{x}_1)\hat{T}^{(x_1^0-x_2^0)/a}\hat{\phi}(0,\mathbf{x}_2)...\hat{T}^{(T/2+x_n^0)/a}\right]}{\mathrm{Tr}[\hat{T}^{T/a}]}.$$

  (The continuum ones discretized in space)

- Prove that $\hat{T}$ is strictly positive. For any $|\Psi\rangle$: $\langle\Psi|\hat{T}|\Psi\rangle > 0, \qquad \langle\Psi|\Psi\rangle = 1$

- Prove that $\hat{T}$ and $\hat{\phi}$ are unique (up to unitary transformations). This is the content of the *reconstruction theorem*

Streater, R. and Wightman, A. S.

All these conditions imply that the quantum Hamiltonian $\hat{H} \equiv -\frac{1}{a} \log \hat{T}$ is self-adjoint and unique.

Alternatively one can invoke the *Osterwalder-Schrader reflection positivity condition* which ensures unitarity as a result of a property of Euclidean correlation functions (i.e. without the need to identify the Hilbert space transfer operator).

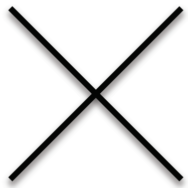
# Lattice Perturbation Theory

Deriving the perturbative expansion and Feynman rules from the lattice theory is completely analogous to the continuum:

$$S^{(1)} = a^4 \sum_x \frac{\lambda}{4!} \phi(x)^4,$$

The Feynman rules for this theory are just like those in the continuum and also the combinatorial factors coming from Wick contractions

$\frac{1}{\hat{p}^2+m^2}$



$-\lambda$

One-loop corrections to the two and four vertex functions:

$$\begin{aligned}
\Gamma^{(2)}(p,-p) &= -(\hat{p}^2+m_0^2) - \frac{\lambda}{2}\int_{BZ}\frac{d^4k}{(2\pi)^4}\frac{1}{\hat{k}^2+m_0^2} \equiv -(\hat{p}^2+m_0^2) - \frac{\lambda}{2} I_1(a,m_0) \\
\Gamma^{(4)}(p_1,p_2,p_3,p_4) &= -\lambda + \left(\frac{\lambda^2}{2}\int_{BZ}\frac{d^4k}{(2\pi)^4}\frac{1}{(\hat{k}^2+m_0^2)(\widehat{k+p_1+p_2})^2+m_0^2} + \mathrm{perm}\right) \\
&\equiv -\lambda + \frac{\lambda^2}{2}\left(I_2(a,m_0,p_1+p_2) + \mathrm{perm.}\right) .
\end{aligned}$$

As in the previous example, all Feynman graphs satisfy the following properties in momentum space:

- periodic functions of all momenta with periodicity $2\pi/a$ in each momentum direction

- loop momenta are integrated only in the $BZ$ and are therefore finite

UV divergences appear as $a \to 0$:

The $\Gamma^{(2)}$ above does not have a finite continuum limit since

$$I_1(a, m_0) = \int_{BZ} \frac{d^4k}{(2\pi)^4} \frac{1}{\hat{k}^2 + m_0^2} = \frac{1}{a^2} F(m_0 a),$$

Expanding $F(x)$ for small $x$:

$$F(0) = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{\sum_\mu (\sin k_\mu/2)^2} = 0.154933...$$

Isolating the log divergence:

$$I_1(a, m_0) = \frac{1}{a^2} F(m_0 a) = \frac{F(0)}{a^2} - m_0^2 \left( -\frac{1}{16\pi^2} \ln(m_0 a)^2 + C + \mathcal{O}(m_0 a)^2 \right),$$

where $C = 0.030345755....$

The UV divergence of $\Gamma^{(2)}$ can be reabsorbed by a redefinition of $m_0^2$

$$\Gamma^{(2)}(p, -p) = -(\hat{p}^2 + m_0^2) - \frac{\lambda}{2} I_1(a, m_0) \equiv -(\hat{p}^2 + m_R^2).$$

The integral $I_2$ is divergent. Expanding in external momenta only the leading order is

$$I_2(a, m_0, 0) \quad = \quad \int_{BZ} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\hat{k}^2 + m_0^2)^2} = -\frac{d}{dm_0^2} I_1(a, m_0)$$

therefore the corresponding divergence can be reabsorbed in $\lambda$:

$$\Gamma^{(4)}(0,0,0,0) = -\lambda + \frac{3\lambda^2}{2} I_2(a, m_0, 0) \equiv -\lambda_R.$$

This is just the usual mass-shell scheme:

$$\Gamma^{(2)}(0,0) = -m_R^2, \quad \left.\frac{d\Gamma^{(2)}(p,-p)}{dp^2}\right|_{p=0} = 1, \quad \Gamma^{(4)}(0,0,0,0) = -\lambda_R.$$

The renormalized quantities are therefore

$$\begin{aligned} m_R^2 &= m_0^2 + \frac{\lambda}{2}\left(\frac{F(0)}{a^2} + \frac{m_0^2}{16\pi^2}\ln(m_0 a)^2 - C m_0^2\right), \\ \lambda_R &= \lambda + \frac{3\lambda^2}{2}\left(-C + \frac{1}{16\pi^2}(\log(m_0 a)^2 + 1)\right). \end{aligned}$$

This can be proven to all orders: challenge, as in the continuum, is to deal with subdivergences

- Prove a power counting theorem to characterize divergent and finite diagrams: Reisz power counting theorem

- Recursive procedure of subtraction: e.g. in the BPHZ subtraction scheme . The $\omega$ of a diagram is reduced by subtracting the Taylor expansion in the external momenta up to order $\omega$, and a forest formula establishes the recursive procedure to subtract subdivergences.

- All-orders proof

Reisz

## Callan-Symanzik equations. Beta functions.

In the Wilsonian RG the effective couplings change smoothly in a way that is locally determined by the effective couplings themselves

Let us consider a fixed $\lambda$ and let us see how $\lambda_R$ changes with $a$ ($m$ is fixed so that $m_R$ is some physical mass) as we approach the continuum limit:

$$\beta(\lambda_R) \equiv a \frac{d\lambda_R}{da}\bigg|_{\lambda} = \frac{3}{(16\pi^2)}\lambda^2 + \mathcal{O}(\lambda^3) = \frac{3}{(16\pi^2)}\lambda_R^2 + \mathcal{O}(\lambda_R^3).$$

This is the Callan-Symanzik beta function.

Integrating

$$\lim_{a\to 0} \lambda_R(a)\bigg|_{\lambda} \sim \lim_{a\to 0} \frac{1}{\log a} = 0,$$

so the continuum theory has a vanishing renormalized coupling, i.e. it is *trivial*.

## Triviality in lattice $\lambda\phi^4$ (and in the SM)

The renormalized coupling decreases as we approach the continuum limit at fixed bare coupling in perturbation theory:

Is the SM a trivial theory $\lambda_R = 0$ ?

*Triviality problem:* the Higgs mass is related to the renormalized coupling in the following way:

$$\frac{m_H^2}{v^2} = \frac{\lambda_R}{3}.$$

Finite $\Lambda \rightarrow$ upper bound on the Higgs mass:

$$\lambda_R^{\max} = \mathrm{Max}\left[\lambda_R(\lambda, \Lambda) | \left\{\lambda \in [0, \infty), \frac{\Lambda}{m_H} \geq 2\right\}\right]$$

This problem can be tackled beyond beyond perturbation theory on the lattice

## Lüscher-Weisz Method

$(m_0, \lambda) \to (\kappa, \bar{\lambda})$:

$$S = a^4 \sum_x \phi(x)^2 + \bar{\lambda}(\phi(x)^2 - 1)^2 - \kappa \sum_\mu (\phi(x)\phi(x+\hat{\mu}) + \phi(x)\phi(x-\hat{\mu}))$$

$$\phi(x) \to \sqrt{2\kappa}\phi(x) \quad m_0^2 \to \frac{1-2\bar{\lambda}}{\kappa} - 8 \quad \lambda \to \frac{6\bar{\lambda}}{\kappa^2}.$$

There is a critical line $\kappa_c(\bar{\lambda})$, where the mass vanishes, where the continuum limit should lie.

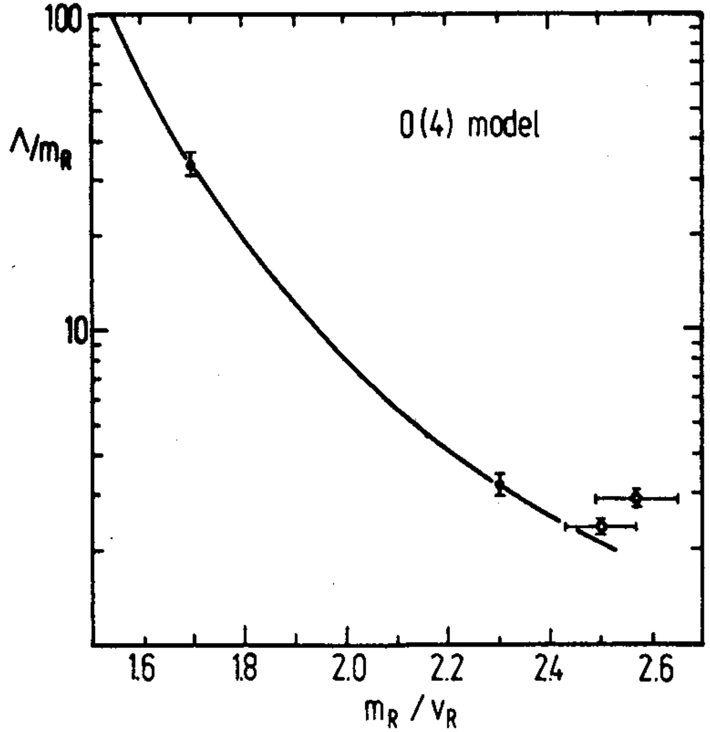
The strategy to study the triviality of the theory follows the following steps:

- $\kappa \ll \kappa_c$ use the hopping parameter expansion to compute $m_R$ and $\lambda_R$ (ok for $m_R a \sim 0.5$)

$$
\begin{aligned}
m_R a &= \frac{1}{\sqrt{\kappa}} \sum_n \alpha_n(\bar{\lambda}) \kappa^n, \\
\lambda_R &= \sum_n \beta_n(\bar{\lambda}) \kappa^n.
\end{aligned}
$$

- Solve the perturbative Callan-Symanzik equations for $\lambda_R$ to approach the critical line with initial conditions given by the results of the hopping expansion

In this way, Lüscher-Weisz could explore the full parameter space $(m_R a)^{-1}$ vs $\lambda_R \sim m_R/v_R$ at $\bar{\lambda} = \infty$:




$$m_H \leq 630GeV,$$

## Free fermions on the lattice

The Euclidean action for free Dirac fermions of mass $m$:

$$S[\psi,\bar{\psi}] = \int d^4x\, \frac{1}{2}\left[\bar{\psi}(x)\gamma_\mu\partial_\mu\psi(x) - \partial_\mu\bar{\psi}(x)\gamma_\mu\psi(x)\right] + m\bar{\psi}(x)\psi(x),$$

where we can choose the *chiral representation* of the $\gamma$ matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix},$$

and the $2\times 2$ matrices are taken to be $e_0 \equiv -I, \quad e_k \equiv -i\sigma_k$:

$$\gamma_\mu^\dagger = \gamma_\mu \quad \{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}.$$

We also define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ satisfying $\gamma_5^\dagger = \gamma_5, \quad \gamma_5^2 = 1$

The classical fermion fields in this action are elements of a Grassmann algebra

$$\{\psi_\alpha(x); \bar{\psi}_\alpha(x)\}_x^{\alpha=1,..4} \leftrightarrow \{c_1, ..., c_n; \bar{c}_1, ...\bar{c}_n\}$$

The number of $c$ and $\bar{c}$ Grassmann variables to represent a general fermion is therefore $4 \times N_{flavour} \times N_{color} \times$ space-time points. The path integral is the Berezin integral over the Grassmann variables

$$\mathcal{Z}_F = \int d\bar{\psi} d\psi e^{-S[\psi,\bar{\psi}]} = \det(\partial\!\!\!/ + m) \quad \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle_F = (\partial\!\!\!/ + m)^{-1}\Big|_{xy}^{\alpha\beta}$$

The Källen-Lehmann representation of the propagator for fermions is:

$$\langle 0|\psi(x)\bar{\psi}(0)|0\rangle_F\Big|_{x_0>0} = \sum_\alpha \int \frac{d^3p}{(2\pi)^3} \, |Z_\alpha|^2 \, \frac{i\gamma_\mu p_\mu - m_\alpha}{2ip_0}\Bigg|_{p_0=iE_{\mathbf{p}}(\alpha)} e^{-E_{\mathbf{p}}(\alpha)x_0} e^{i\mathbf{p}\mathbf{x}}$$

with $E_{\mathbf{p}}(\alpha) = \sqrt{m_\alpha^2 + \mathbf{p}^2}$ and $Z_\alpha \equiv \langle 0|\Psi(0)|\alpha\rangle$.

## Naive fermions and doubling

The fields are defined at the lattice points only and the derivatives are substituted by their discrete versions:

$$S[\psi,\bar{\psi}] = a^4 \sum_{x,\alpha,\mu} \bar{\psi}_\alpha(x) \left[\frac{1}{2}(\hat{\partial}_\mu + \hat{\partial}^*_\mu) + m\right] \psi_\alpha(x) = a^4 \sum_{x,y} \bar{\psi}_\alpha(x) K^{\alpha\beta}_{xy} \psi_\beta(y),$$

$$K^{\alpha\beta}_{xy} \equiv \sum_\mu \frac{1}{2a} (\gamma_\mu)_{\alpha\beta} \left(\delta_{yx+a\hat{\mu}} - \delta_{yx-a\hat{\mu}}\right) + m\delta_{\alpha\beta}\delta_{xy}.$$

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_F = \frac{1}{a^4}\left(K^{-1}\right)^{\alpha\beta}_{xy}.$$

$$\tilde{K}^{\alpha\beta}_{pq} = a^4 \left[ \sum_\mu \frac{i}{a} \gamma_\mu \sin(q_\mu a) + m \right]_{\alpha\beta} (2\pi)^4 \delta_P(p+q),$$

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_F = \int_{BZ} \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m}.$$

The integral over $p_0$ as a sum of residues of single poles in the band $|\mathrm{Re} p_0| \leq \pi/a$ and $\mathrm{Im} p_0 \geq 0$:

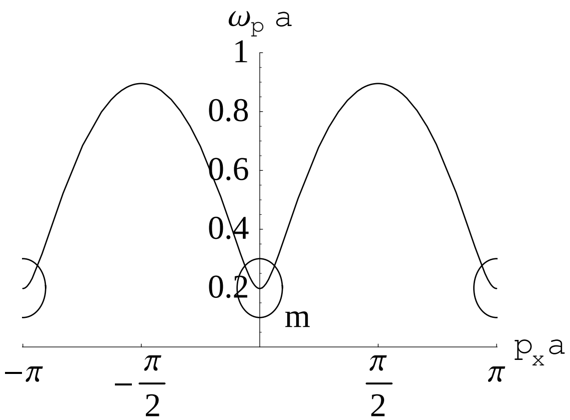
$$e^{ip_0 a} = \pm e^{-\omega_{\mathbf{p}} a} \equiv \pm \left( \sqrt{1 + M_{\mathbf{p}}^2} - M_{\mathbf{p}} \right)$$

$$M_{\mathbf{p}}^2 \equiv m^2 a^2 + \sum_{k=1}^{3} \sin(p_k a)^2.$$

$$
\begin{aligned}
\langle \psi_\alpha(x)\bar{\psi}_\beta(0)\rangle_F &= \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\mathbf{x}}e^{-\omega_{\mathbf{p}}x_0}}{\sinh(2\ \omega_{\mathbf{p}}a)}\Bigg[\Bigg(\gamma_0 \sinh\omega_{\mathbf{p}}a - i\sum_k \gamma_k \sin p_k a + ma\Bigg) \\
&+ (-1)^{x_0/a}\Bigg(-\gamma_0\sinh\omega_{\mathbf{p}}a - i\sum_k \gamma_k \sin p_k a + ma\Bigg)\Bigg].
\end{aligned}
$$

Two new features appear with respect to the scalar case:

- there are two terms in the sum with the same energy, $\omega_{\mathbf{p}}$, but different residue
- the energy, $\omega_{\mathbf{p}}$, has minima at $p_k = \bar{p}_k \equiv n_k \frac{\pi}{a} \quad n_k = 0, 1.$




$$\lim_{a\to 0}\ \omega_{\mathbf{p}}\big|_{p_k = n_k\pi/a} = m.$$

Near the continuum limit, it is justified to consider the contribution near these minima

$$p_j = \bar{p}_j^{(i)} + k_j, \quad k_j a \ll 1 \quad j = 1, ..., 2^3$$

$$\begin{aligned}
&\langle \psi_\alpha(x) \bar{\psi}_\beta(0) \rangle_F \\
= \quad & \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{kx}} e^{-\omega_{\mathbf{p}} t}}{2k_0} \left[ \gamma_0 \cos(\bar{p}_0^{(\alpha)} a) k_0 - i \sum_j \gamma_j \cos(\bar{p}_j^{(\alpha)} a) k_j + m \right],
\end{aligned}$$

16 terms: $\bar{p}_\mu^{(\alpha)} = (n_0^{(\alpha)}, n_1^{(\alpha)}, n_2^{(\alpha)}, n_3^{(\alpha)}) \frac{\pi}{a}, \quad n_\mu^{(\alpha)} = 0, 1$

Define unitary operators

$$S_\alpha \equiv \prod_\mu (i\gamma_\mu \gamma_5)^{n_\mu^{(\alpha)}} \quad S_\alpha \gamma_\mu S_\alpha^\dagger = \gamma_\mu \cos(\bar{p}_\mu^{(\alpha)} a)$$

The propagator is then

$$\sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{kx}} e^{-\omega_{\mathbf{p}} t}}{2k_0} S_\alpha \left[ \left( \gamma_0 k_0 - i \sum_k \gamma_k k_k + m \right) \right] S_\alpha^{-1}$$

Each term is the contribution of a relativistic fermion in the continuum, since $S_\alpha$ is just a similarity transformation: an equivalent representation of the $\gamma$ matrices.

16 $(2^d)$ relativistic free fermions instead of 1.... this is the famous *doubling problem*

## Doubling and chiral symmetry

There is a deep connection between the doubling problem and the difficulty to regularize chirality

The continuum free fermion action for $m = 0$ has a global symmetry under chiral rotations:

$$\psi(x) \to e^{i\alpha\gamma_5}\psi(x) \qquad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\alpha\gamma_5}$$

We can consider a free Weyl fermion as the left or right chiral component:

$$\psi_L \equiv \frac{1-\gamma_5}{2}\psi, \qquad \psi_R \equiv \frac{1+\gamma_5}{2}\psi$$

Let us see what happens when we naively discretize the action of a Weyl fermion.

The naive propagator is (for $m = 0$):

$$\langle \psi_L(x)\bar{\psi}_L(0)\rangle_F = \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{kx}}e^{-\omega_\mathbf{p}t}}{2k_0} S_\alpha \left(\gamma_0 k_0 - i\sum_k \gamma_k k_k\right) S_\alpha^{-1}\left(\frac{1-\gamma_5}{2}\right)$$
$$= \sum_{\alpha=1}^{16} e^{i\bar{p}^{(\alpha)}x} \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{kx}}e^{-\omega_\mathbf{p}t}}{2k_0} S_\alpha \left[\left(\gamma_0 k_0 - i\sum_k \gamma_k k_k\right)\left(\frac{1-\epsilon^a\gamma_5}{2}\right)\right] S_\alpha^{-1}$$

where $\epsilon^\alpha = (-1)^{\sum_\mu n_\mu^{(\alpha)}}$. Therefore each of the doublers contribute either a left-handed relativistic Weyl fermion for $\epsilon^\alpha = 1$ or a right-handed one $\epsilon^\alpha = -1$ in the continuum. It turns out that the number of right and left movers is the same!

$$\begin{aligned} \text{Left} &: \quad 1+6+1 = 8 \\ \text{Right} &: \quad 4+4 = 8 \end{aligned}$$

This result can be generalized to rather arbitrary forms of the fermionic action: *Nielsen-Ninomiya theorem*

$$S_F = a^4 \sum_{x,y} \bar{\psi}(x)\gamma_\mu F_\mu(x-y)(1-\gamma_5)\psi(y)$$

satisfying the following properties:

- Action quadratic in the fermion fields
- Invariant under lattice translations (i.e. diagonal in momentum space)
- Local (smooth Fourier transform)
- Hermitian action: $F_\mu(x)^* = -F_\mu(x)$ (implies a real Fourier transform of $F_\mu$ field)

The FT $F_\mu(p)$ must have some isolated zeros (in order to have a continuum limit). Let us call $\bar{p}^\alpha$ the zeros of $F_\mu(p)$. Sufficiently close we can approximate

$$F_\mu(p) \simeq M^{(\alpha)}_{\mu\nu}(p - \bar{p}^\alpha)_\nu + ... = M^{(\alpha)}_{\mu\nu} k^{(\alpha)}_\nu + ...,$$

$M^{(\alpha)}_{\mu\nu}$ is a real matrix that can be decomposed in general as

$$M^{(\alpha)}_{\mu\nu} = O^{(\alpha)}_{\mu\rho} S^{(\alpha)}_{\rho\nu},$$

$O^{(\alpha)}$ is an orthogonal matrix and $S^{(\alpha)}$ is a positive real symmetric matrix

Consider an $SO(d+1)$ rotation in $d+1$ (with $d$ even):

$$\begin{pmatrix} O^{(\alpha)} & 0 \\ 0 & (\det O^{(\alpha)})^{-1} \end{pmatrix}$$

The spinor representation of $SO(d+1)$ are the $d+1$ $\gamma$ matrices: $(\gamma_\mu, \gamma_5)$. There must exist therefore a unitary matrix that implements the rotation in the spinor representation such that

$$\Lambda^{(\alpha)}\gamma_\nu {\Lambda^{(\alpha)}}^{-1} = O^{(\alpha)}_{\mu\nu}\gamma_\mu \quad \Lambda^{(\alpha)}\gamma_5 {\Lambda^{(\alpha)}}^{-1} = \det{}^{-1} O^{(\alpha)}\gamma_5$$

Therefore we can rewrite the action as

$$\sum_\alpha \int \frac{d^4k^{(\alpha)}}{(2\pi)^4}\bar{\psi}(-k^{(\alpha)})\Lambda^{(\alpha)}\gamma_\rho S^{(\alpha)}_{\rho\nu} k^{(\alpha)}_\nu (1-\det O^{(\alpha)}\gamma_5){\Lambda^{(\alpha)}}^{-1}\psi(k^{(\alpha)})$$

The real positive matrix $S^{(\alpha)}$ is harmless and can be reabsorbed in a rescaling of the momentum. However we see that there are left-movers and right-movers depending on the sign of $\det O^{(\alpha)}$.

*Poincaré-Hopf theorem* states:

$$\sum_\alpha \det O^{(\alpha)} = \text{ Euler characteristic of the manifold where } F_\mu(p) \text{ isdefined}$$

Euler characteristic (Brillouin zone) $= 0$: there must be as many zeros with $\det O^{(\alpha)} = 1$ as those with $\det O^{(\alpha)} = -1$

(This is a generalization of a simpler version of the theorem for one dimensional functions: a smooth and periodic function that crosses zero must do it an even number of times with opposite signs of the derivatives at the zeros)

Not surprisingly the easiest way to get rid of doublers is to break chiral symmetry: Wilson's solution to the doubling problem.

## Wilson fermions

K. Wilson proposed to add to the naive action the following term

$$\Delta_W S = -a^4 \sum_x \bar{\psi}(x) \frac{ra}{2} \hat{\partial}^*_\mu \hat{\partial}_\mu \psi(x)$$

where $r$ is some arbitrary constant of $O(1)$. It is easy to see that the propagator in momentum space is modified to

$$\langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_F = \int_{BZ} \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{\sum_\mu i\gamma_\mu \frac{\sin(p_\mu a)}{a} + m + \frac{r}{a} \sum_\mu (1 - \cos p_\mu a)}$$

As before the integration over $p_0$ can be performed as a sum of residues of the solutions, in the region $\mathrm{Im}\, p_0 > 0$, $-\pi < \mathrm{Re}\, p_0 < \pi$, of

$$\sum_\mu \sin^2 p_\mu + \left( m + \frac{r}{a} \sum_\mu (1 - \cos p_\mu a) \right)^2 = 0$$

For $r = 1$ (Wilson's choice) the *only* solution is at $p_0 = i\omega_{\mathbf{p}}$ satisfying

$$\cosh \omega_{\mathbf{p}} = \frac{1 + \sum_k \sin^2 p_k a + (ma + 1 + \sum_k (1 - \cos p_k a))^2}{2(ma + 1 + \sum_k (1 - \cos p_k a)}$$

The energy at $\bar{p}_k^{(\alpha)} = n_k^{(\alpha)} \frac{\pi}{a}$

$$\omega_{\mathbf{p}}^{(\alpha)} = \frac{1}{a} \log \left( 1 + ma + 2 \sum_k n_k^{(\alpha)} \right),$$

the only pole that survives in the continuum limit (i.e. $\lim_{a \to 0} a\omega_{\mathbf{p}} = m$) corresponds to $n_k^{(\alpha)} = 0$ for all $k$. The others have energies $\sim a^{-1}$

## Transfer matrix of Wilson fermions and unitarity

Actually, it is for Wilson fermions with $r = 1$ the only fermion regularization for which the transfer matrix has been proven to be positive so that the lattice formulation has a Hilbert space representation !

The proof relies on the explicit construction of the transfer operator $\hat{T}$ acting on Fock space such that

$$\mathcal{Z}_F = \lim_{N\to\infty} \mathrm{Tr}[\hat{T}^N]$$

and proving that it is positive in such a way that the Hamiltonian $\hat{H} = -\frac{1}{a}\log \hat{T}$ is well defined.

Wilson; Lüscher

The case $r \neq 1$ cannot be treated in the same way. Reflection positivity on the other hand can be proven for $r \leq 1$

## Kogut-Susskind or staggered fermions

Basic idea: use some of the $2^d$ doublers to represent some of the $2^{d/2}$ spinor components: $2^d/2^{d/2}$ replicas instead of $2^d$

The advantage is that the lattice action can be shown to have an extra exact $U(1)$ chiral symmetry compared to the Wilson action.

1) Diagonalize the action in spinor space: find a unitary $S_x$

$$\psi(x)_\alpha \to (S_x)_{\alpha\beta}\psi_\beta(x) \equiv \chi_\alpha(x)$$

such that the naive action is:

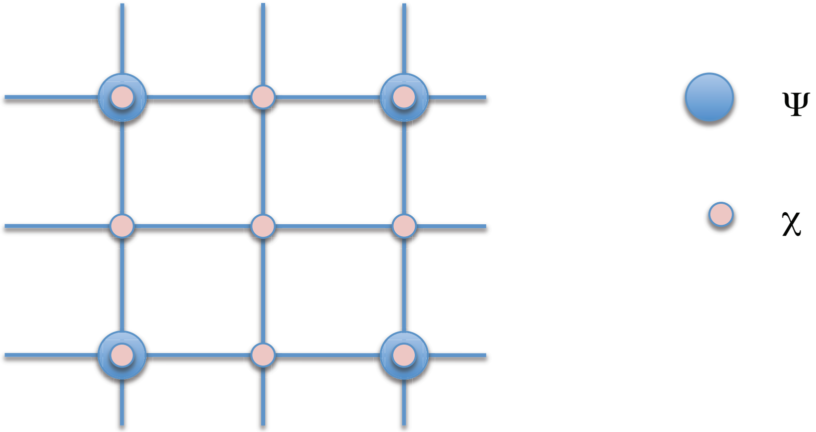
$$S_{KS} = a^4 \sum_{x,\alpha} \left[ \sum_\mu \rho_{x\mu} \bar{\chi}^\alpha(x) \frac{1}{2} \left(\chi^\alpha(x+a\hat{\mu}) - \chi^\alpha(x-a\hat{\mu})\right) + m\bar{\chi}^\alpha(x)\chi^\alpha(x) \right]$$

For example:

$$S_x \equiv \gamma_0^{n_0} ... \gamma_3^{n_3} = \prod_\mu \gamma_\mu^{n_\mu} \quad x = a\ (n_0, n_1, n_2, n_3) \quad \rho_{x\mu} = (-1)^{\sum_{\rho<\mu} n_\rho}$$

We can therefore consider just $one$ of this replicas that we call $\chi$.

2) Reconstruction of the Dirac field $\Psi$ defined on a coarser lattice with a doubled lattice spacing $2a$:




$$x = an_\mu = 2aN_\mu + az_\mu, \quad z_\mu = 0, 1$$

$$\chi(na) \equiv \psi_z(2Na) \quad \Psi^{\alpha i}(2Na) \equiv \sum_z (S_z)_{\alpha i} \psi_z(2Na) \quad S_z \equiv \prod_\nu \gamma_\nu^{z_\nu}$$

Four flavour theory with no doublers

$$S_{KS} = (2a)^4 \sum_{\mu,N} \left[ \bar{\Psi}(N)(\gamma_\mu \otimes 1)\frac{1}{2}(\hat{\partial}_\mu + \hat{\partial}^*_\mu)\Psi(N) + a\bar{\Psi}(N)(\gamma_5 \otimes \gamma_\mu^T \gamma_5^T)\frac{1}{2}a\ \hat{\partial}_\mu \hat{\partial}^*_\mu \Psi(N) \right]$$
$$+ (2a)^4 m \sum_N \bar{\Psi}(N)\Psi(N),$$

- The action looks quite close to the Wilson action. The difference is the Dirac/flavour structure of the Wilson term.
- The action has an exact $U(1)$ chiral symmetry for $m = 0$ under spin-flavour rotations of the form

$$\Psi_N \to e^{i\alpha(\gamma_5 \otimes \gamma_5^T)}\Psi_N, \quad \bar{\Psi}_N \to \bar{\Psi}_N e^{i\alpha(\gamma_5 \otimes \gamma_5^T)}$$

- Staggered fermions perfectly ok to describe $N_f = 4$ degenerate quarks, but the use for $N_f = 1$ is questionable (strong division in the community...)

**Ginsparg-Wilson fermions** Ginsparg-Wilson investigated the remant of chiral symmetry in RG blocked actions:

$$\chi(na) = \int d^4y \; \omega_n(y)\psi(y)$$

$$\begin{aligned} \exp(-S_a(\bar{\chi},\chi)) &= \int D\bar{\psi}D\psi \exp\Big[-S(\bar{\psi},\psi) \\ &- \sum_{nm}\left(\bar{\chi}_n - \int d^4x\bar{\psi}(x)\omega_n^\dagger(x)\right)\frac{1}{2}\delta_{nm}\left(\chi_m - \int d^4y\omega_m(y)\psi(y)\right)\Big], \end{aligned}$$

$$S_a = \bar{\chi}_n D_{nm}\chi_m$$

where $D$ is a complicated operator. If we do an infinitesimal chiral rotation of the blocked fields:

$$\chi \to e^{i\epsilon\gamma_5}\chi \quad \bar{\chi} \to \bar{\chi}e^{i\epsilon\gamma_5}$$

The following relation holds:

$$\{\gamma_5, D\} = aD\gamma_5 D \quad \{\gamma_5, D^{-1}\} = a\gamma_5$$

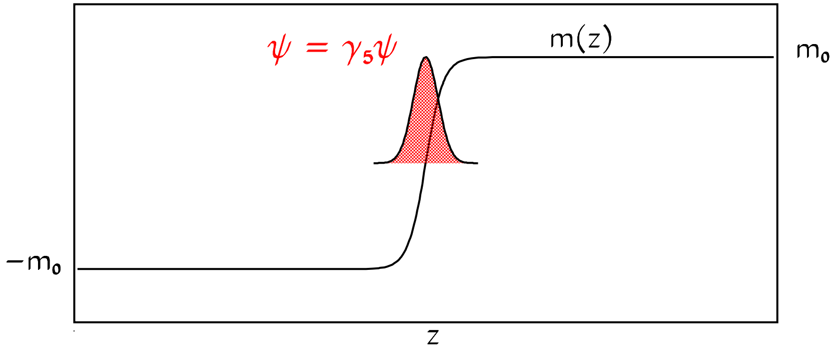
" We have unfortunately not yet found either (8a) or (8b) to yield any tractable gauge-invariant solution. "

Ginsparg-Wilson

Hasenfratz rediscovered it in 1997 and realized that the classically perfect Dirac operator satisfies it. But is not an explicit construction either...

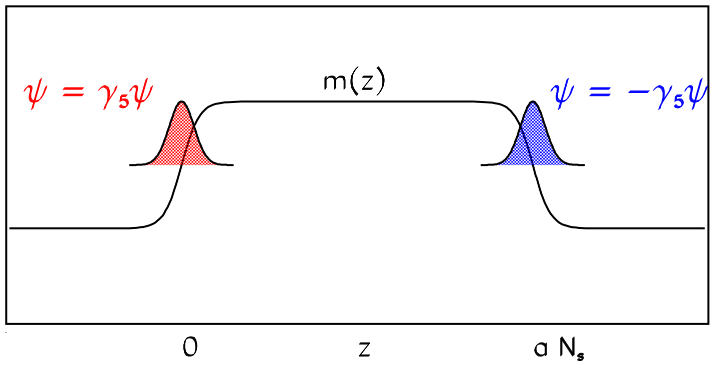
# Domain-wall in 5D

Rubakov, Shaposhnikov; Callan, Harvey




On the lattice the DW construction $a$ , $a_s$, $N_s$ :

Kaplan; Shamir



The effective action of the light boundary fields can be described in terms of 4D operator $aD_{N_s}$, such that $\lim_{N_s\to\infty} aD_{N_s}$ satisfies the GW relation!

$$\lim_{a_s\to 0, N_s\to\infty} aD_{N_s} = aD_{ov} = 1 - \gamma_5 \mathrm{sign}(\mathrm{Q}) \qquad \mathrm{Q} \equiv \gamma_5(\mathrm{m_0} - \mathrm{D_W})$$

Neuberger

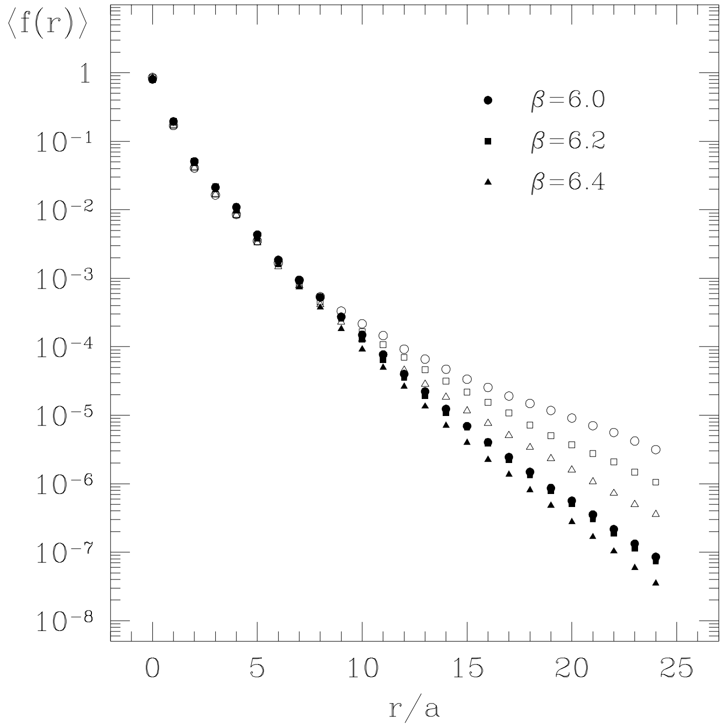
It has the right continuum limit, no doublers, and an exact lattice chiral symmetry

$$\delta_\chi \Psi = \epsilon\, \gamma_5(1 - aD)\Psi \qquad \delta_\chi \bar{\Psi} = \epsilon \bar{\Psi}\gamma_5 \quad \to \delta_\chi S_f = 0$$

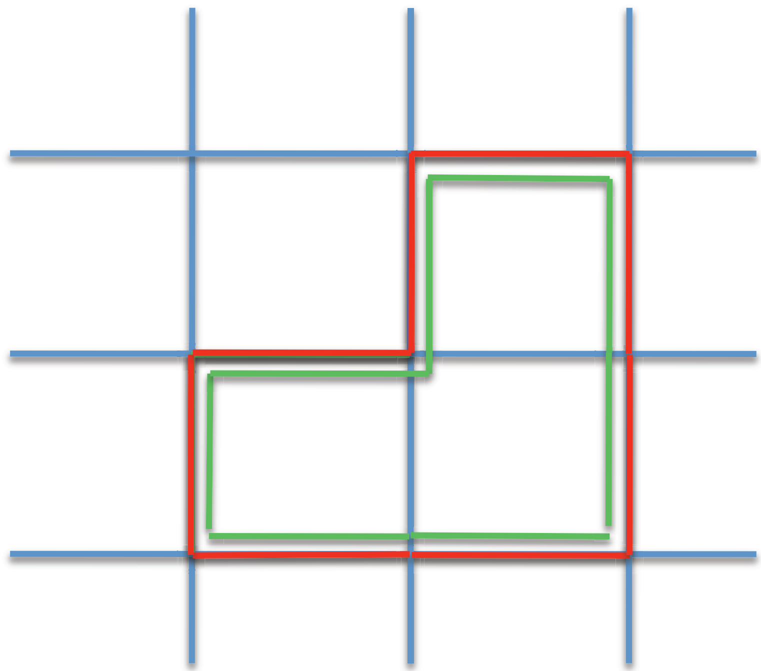
Lüscher

It is also a local operator $||D_{ov}(0,r)|| \leq e^{-\gamma|r|/a}$

PH., Jansen, Lüscher



# Lecture III: Gauge Fields on the Lattice



K. Wilson (and J. Smit) figured out how to formulate a quantum field theory of gauge fields on the lattice preserving an exact gauge invariance

**Abelian case**

Gauge invariance in continuum:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu \Lambda(x), \\ \phi(x) &\to e^{iq\Lambda(x)}\phi(x) \equiv \Omega(x)\phi(x) \end{aligned}$$

If the electromagnetic field strength vanishes in all space, we can choose a gauge $A_\mu = 0$:

$$\begin{aligned} S &= \frac{a^4}{2}\sum_{x,y}\phi^\dagger(x)K_{xy}\phi(y), \\ K_{xy} &= -\frac{1}{a^2}\sum_{\hat{\mu}}\left(\delta_{x+a\hat{\mu}y} + \delta_{x-a\hat{\mu}y} - 2\delta_{xy}\right) + m^2\delta_{xy} \end{aligned}$$

If we change the gauge

$$\phi(x) \to e^{iq\Lambda(x)}\phi(x) = \phi'(x), \quad \phi(x)^\dagger \to \phi(x)^\dagger e^{-iq\Lambda(x)} = \phi'(x)^\dagger, \quad A'_\mu = \partial_\mu \Lambda(x)$$

The action in terms of the prime fields, $\phi'(x), A'_\mu(x)$ is

$$S \;=\; \frac{a^4}{2}\sum_{x,y}\phi'^\dagger(x)K^\Lambda_{xy}\phi'(y),$$

$$K^\Lambda_{xy} = -\frac{1}{a^2}\sum_{\hat{\mu}}\left(\delta_{x+a\hat{\mu}y}U_\mu(x) + \delta_{x-a\hat{\mu}y}U^\dagger_\mu(x-a\hat{\mu}) - 2\delta_{xy}\right) + m^2\delta_{xy},$$

$$U_\mu(x) \equiv e^{iq\Lambda(x)}e^{-iq\Lambda(x+a\hat{\mu})} = e^{-iq\int_x^{x+a\hat{\mu}}\partial_\mu\Lambda(x)dx_\mu} \equiv e^{-iq\int_x^{x+a\hat{\mu}}dx_\mu A'_\mu(x)}$$

The *link variable* is

$$U_\mu(x) \equiv \exp\left(iq\int_{x+a\hat{\mu}}^{x} A'_\mu(x)dx_\nu\right)$$

For any gauge field, the action is gauge invariant:

$$\begin{aligned} U_\mu^\Lambda(x) &= \exp\left(i\int_{x+a\hat{\mu}}^{x}(A_\mu + \partial_\mu\Lambda)dx_\mu\right) = \exp\left(i\int_{x+a\hat{\mu}}^{x} A_\mu dx_\mu + i\Lambda(x) - i\Lambda(x+\hat{\mu})\right) \\ &= \Omega(x)U_\mu(x)\Omega^\dagger(x+a\hat{\mu}) \end{aligned}$$

The lattice scalar-gauge action is indeed invariant under the gauge transformation

$$\phi'(x) \to \Omega(x)\phi'(x) \quad U_\mu(x) \to \Omega(x)U_\mu(x)\Omega^\dagger(x+a\hat{\mu})$$

It is easy to generalize this procedure to fermions or any other charged fields: change lattice partial derivatives to covariant ones:

$$
\begin{aligned}
\hat{\partial}_\mu \psi(x) &= \frac{1}{a}\left(\psi(x+a\hat{\mu}) - \psi(x)\right) \to \nabla_\mu \psi(x) = \frac{1}{a}\left(U_\mu(x)\psi(x+a\hat{\mu}) - \psi(x)\right), \\
\hat{\partial}^*_\mu \psi(x) &= \frac{1}{a}\left(\psi(x) - \psi(x-a\hat{\mu})\right) \to \nabla^*_\mu \psi(x) = \frac{1}{a}\left(\psi(x) - U^\dagger_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})\right)
\end{aligned}
$$

In the continuum

$$\mathcal{Z} = \int dA_\mu e^{-S[A_\mu]} \quad S[A_\mu] \equiv \frac{1}{4}\int d^4x F_{\mu\nu}F_{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength.

*Wilson loop*: any ordered closed loop of link variables

$$W(x) \to \Omega(x)W(x)\Omega(x)^\dagger$$

In the case of an abelian group, $W(x)$ is therefore invariant.

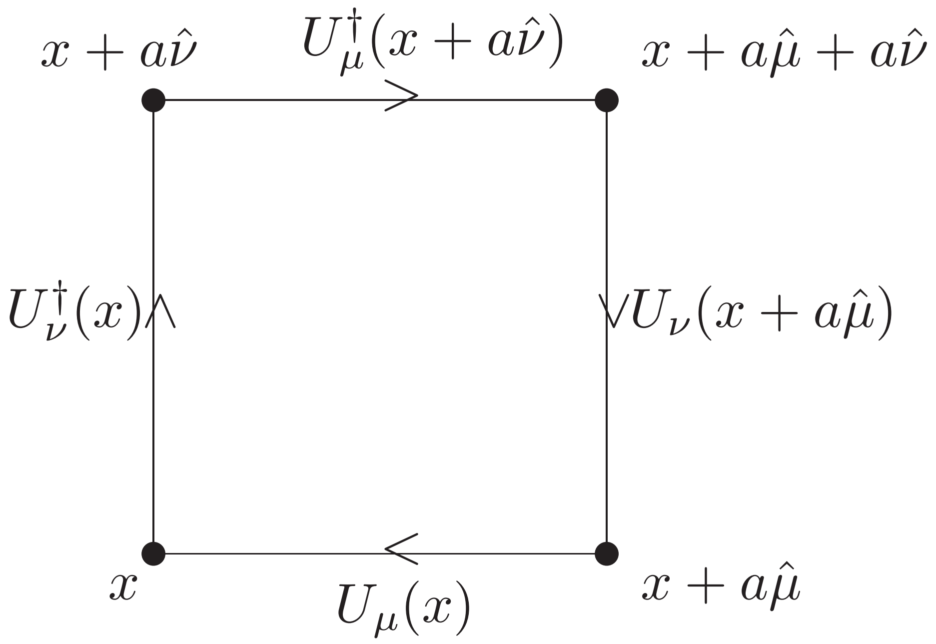



Figure 1: Plaquette

Since we want our action to be local, we can try with the smallest Wilson loop: *plaquette*

$$U_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x + a\hat{\mu})U^\dagger_\mu(x + a\hat{\nu})U^\dagger_\mu(x)$$

A good lattice action for the link variables:

$$S[U] = \frac{1}{q^2} \sum_x \sum_{\mu \leq \nu} \left[ 1 - \frac{1}{2} \left( U_{\mu\nu}(x) + U^{\dagger}_{\mu\nu}(x) \right) \right]$$

- Local
- Real
- Gauge invariant
- Right classical continuum limit: $\lim_{a \to 0} S[U] = \int d^4x \frac{1}{4} F^2_{\mu\nu} + \mathcal{O}(a^2)$

We still need to define the measure over the link variables. The link variables are elements of $U(1)$:

$$dU \equiv \prod_{\mu,x} d\phi_\mu(x) \quad U_\mu(x) = e^{i\phi_\mu(x)}, \quad 0 \leq \phi_\mu(x) \leq 2\pi.$$

This measure is gauge invariant:

$$\phi_\mu(x) \to \phi'_\mu(x) = \Lambda(x) + \phi_\mu(x) - \Lambda(x + a\hat{\mu}), \quad d\phi_\mu(x) = d\phi'_\mu(x)$$

## Non-abelian case

The vector gauge potential $A_\mu(x)$ in a $SU(N)$ Yang-Mills theory takes values in the Lie algebra of the gauge group:

$$A_\mu(x) = A^a_\mu(x) T^a,$$

where the coefficients $A^a_\mu(x)$ are real and $T^a = (T^a)^\dagger$ are the hermitian generators of the algebra.

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)],$$

is also an element of the algebra.

A gauge transformation is:

$$A_\mu(x) \to \Omega(x) A_\mu(x) \Omega(x)^{-1} + i\Omega(x)\partial_\mu \Omega(x)^{-1},$$

where $\Omega(x) \in SU(N)$.

It implies the following transformation of the field tensor

$$F_{\mu\nu}(x) \to \Omega(x) F_{\mu\nu}(x) \Omega(x)^{-1}.$$

The Euclidean Yang-Mills action is given by

$$S[A_\mu] = \frac{1}{2g_0^2} \int d^4x \mathrm{Tr}\left[F_{\mu\nu} F_{\mu\nu}\right],$$

and is therefore gauge invariant.

A colored scalar field in the fundamental representation of this symmetry group transforms as

$$\phi(x) \to \phi'(x) = \Omega(x)\phi(x) \quad \Omega \in SU(N)$$

Start with the free action of colored scalar fields and identify the way gauge fields appear in the lattice action by performing a gauge transformation of the coloured fields, $\phi(x) \to \phi'(x)$ that will then be coupled to

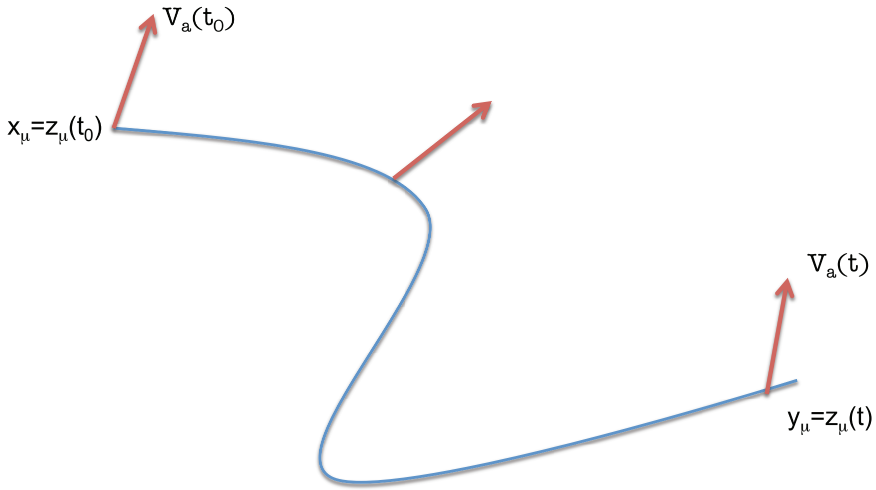
$$A'_\mu(x) = i\Omega(x)\partial_\mu\Omega(x)^{-1}$$

The same result as in the $U(1)$ case with the link variables:

$$U_\mu(x) \equiv \Omega(x)\Omega(x + a\hat{\mu})^\dagger,$$

This is a parallel transporter of the non-abelian gauge field from $x + a\hat{\mu}$ to $x$.

## *Parallel transporter for $SU(N)$*




$v_a(t_0) \to N$ component vector of unit length $\quad z_\mu(t) \to$ a curve in $R^4$

A parallel transport of $v_a$ from points $t_0$ to $t$, in the presence of the field $A_\mu$ is the solution of

$$\left[\frac{d}{dt} - i\frac{dz_\mu(t)}{dt}A_\mu(z_\mu(t))\right]\mathbf{v}(t) = 0$$

The parallel transporter from $z_\mu(t_0) = x_\mu$ to $z_\mu(t) = y_\mu$, $P(y,x)$, is

$$v_a(t) = P_{ab}(y,x)v_b(t_0)$$

The solution can be written as a series in $A_\mu$:

$$\begin{aligned}
\mathbf{v}(t) &= \Big(I + i\int_0^t dt_1 \dot{z}_\mu(t_1)A_\mu(z(t_1)) \\
&- \int_0^t dt_1 \dot{z}_\mu(t_1)A_\mu(z(t_1))\int_0^{t_1} dt_2 \dot{z}_\nu(t_2)A_\nu(z(t_2)) + ...\Big)\mathbf{v}(t_0) \\
&\equiv P\exp\left(i\int_x^y A_\mu(z)dz_\mu\right)\mathbf{v}(t_0)
\end{aligned}$$

## Plaquette action

These properties are sufficient to ensure the gauge invariance of the *plaquette action* also for $SU(N)$:

$$S[U] \equiv C \sum_x \sum_{\mu<\nu} \mathrm{Tr} \left[ 1 - \frac{1}{2} \left( U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x) \right) \right]$$

The coefficient $C$ can be chosen to recover the conventional normalization in the classical continuum limit:

$$C \equiv \frac{2}{g_0^2}$$

A gauge invariant measure for the link variables is the de Haar measure on the group, which obeys two essential properties

- it is gauge invariant. For any $V \in SU(N)$

$$\int_{SU(N)} dU f(U) = \int_{SU(N)} f(VU) dU = \int_{SU(N)} f(UV) dU$$

- it is normalized

$$\int_{SU(N)} dU = 1$$

- it is unique. For any parametrization of the group in terms of $n$ coordinates, $z_i$, then

$$dU = w(z) dz_1 dz_2 ... dz_n$$

To find $w(z)$, we can define a metric tensor in the group by

$$g_{kl} \equiv -2\mathrm{Tr}[U\partial_k U^{-1})(U\partial_l U^{-1})]$$

which can be shown to be positive definite and gauge invariant.

The measure in this coordinates is

$$w(z) = c\sqrt{\det g(z)},$$

where $c$ is obtained from the normalization condition.

Two observations are in order:

- the integrals over the link variables are finite, there is no need to gauge fix

- the action is real and positive definite

$$S[U] \sim \sum_P \mathrm{Tr}[2 - U_P - U_P^\dagger] = \sum_P \mathrm{Tr}[(1 - U_P)(1 - U_P^\dagger)] \geq 0,$$

the equality being obtained only when all plaquettes are unity: $U_P = 1$.

An important question is whether there is unitarity in this theory, for which we should make contact with the operator formulation via the transfer matrix.

## Transfer matrix and unitarity of the plaquette action

Wilson; Lüscher

- Identify field operators $t = 0$ that represent the creation of particles: spatial plaquettes

$$\hat{U}_{kl}(\mathbf{x}, 0) \quad k = 1, 2, 3$$

- Identify a transfer operator that

$$\mathcal{Z} = \lim_{N \to \infty} \mathrm{Tr}[\hat{T}^N], \qquad N = T/a$$

- Prove that the transfer operator is positive

All this can be done! Unitarity or Hilbert space representation of the lattice formulation.

Having a unitary theory is reassuring, but the infrared behaviour of this theory is highly non-trivial. We believe two fundamental phenomena take place:

- Generation of a mass gap (in spite of the absence of dimensionful couplings)
- Confinement: asymptotic states are gauge singlets

A very useful intuition can be obtained from the strong coupling expansion of the lattice theory, as first realized by Wilson, where both phenomena can be shown to take place.

## Strong Coupling Expansion

The strong coupling expansion is an expansion in inverse powers of the coupling $g_0$, which by the structure of the path integral is equivalent to a large temperature expansion of the statistical system:

$$\mathcal{Z} = C \int \prod_l dU_l e^{-\frac{\beta}{2N}\sum_p \left[\mathrm{Tr}[U_p] + \mathrm{Tr}[U_p^\dagger]\right]}$$

where $\beta \equiv \frac{2N}{g_0^2}$

The large $g_0$ expansion is a Taylor expansion in $\beta$ (large temperature):

$$\mathcal{Z} = \int \prod_l dU_l \prod_p \sum_n \frac{1}{n!} \left(\frac{\beta}{2N}\right)^n (\chi(U_p) + \chi(U_p)^*)^n$$

## Leading order

- Has the lowest number of plaquettes.
- All link variables must be shared by at least two plaquettes, since

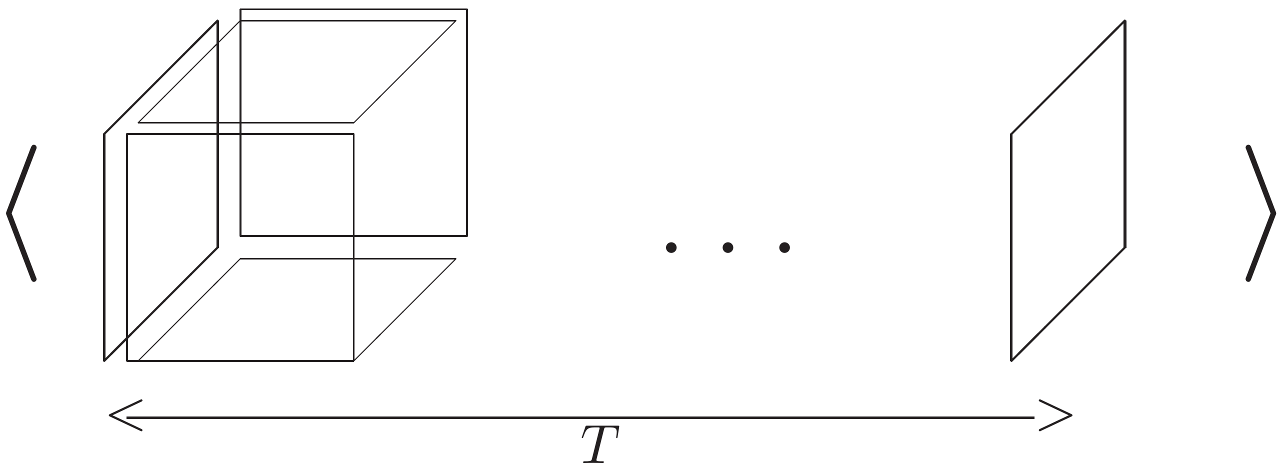
$$\int dU \ U_{\alpha\beta} = 0$$

For the two examples following the only non-trivial integral is that of two links

$$\int dU \ U_{\alpha\beta} U^{\dagger}_{\gamma\delta} = \frac{1}{N} \delta_{\alpha\delta} \delta_{\beta\gamma}$$

## Plaquette-Plaquette correlator and mass gap

Correlation functions of spatial plaquettes describe the propagation and scattering of physical particles: *glueballs*. Mass gap shows up in the correlator of spatial plaquettes at large time separation.




Each internal plaquette brings a factor $\beta/2N$, each integral over two paired links brings in a factor $1/N$ and each vertex gives a factor of $N$:

$$C_{pp}(T) \sim \left(\frac{\beta}{2N}\right)^{N_p} \left(\frac{1}{N}\right)^{N_i} N^{N_v}$$

$$
\begin{aligned}
N_p &= \#plaquettes = 4T/a \\
N_i &= \#integrals = \#links/2 = 2(N_p + 2) \\
N_v &= \#vertices = N_v = 4(T/a + 1)
\end{aligned}
$$

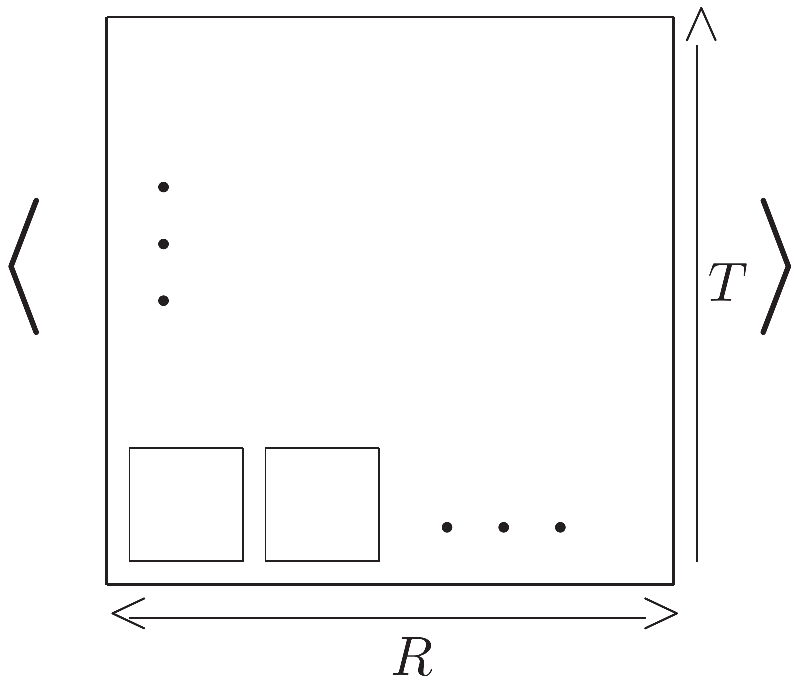
$$
C_{pp}(T) \sim \left(\frac{\beta}{2N^2}\right)^{4T/a} = \exp\left(-\frac{4}{a}\log\left(\frac{2N^2}{\beta}\right)T\right)
$$

the correlator decays exponentially in time as expected in a theory with a finite mass gap !

$$
m \sim \frac{4}{a}\log\left(\frac{2N^2}{\beta}\right)
$$

Unfortunately no continuum limit can be reached in the strong coupling expansion since $\lim_{a\to 0} ma = \text{finite}.$

Let us consider a rectangular loop with two spatial sides and two temporal ones, $W_{RT}$:




At LO in strong coupling:

$$\langle W_{RT} \rangle = \left( \frac{\beta}{2N} \right)^{N_p} \left( \frac{1}{N} \right)^{N_i} N^{N_v}, \quad N > 2$$

$$N_p = (R/a)(T/a) \quad N_i = 2N_p + (R/a + T/a) \quad N_v = (R/a+1)(T/a+1)$$

Therefore

$$\langle W_{RT} \rangle \sim N \left( \frac{\beta}{2N^2} \right)^{RT/a^2} \sim \exp\left( -\log\left( \frac{2N^2}{\beta} \right) \frac{RT}{a^2} \right) \sim \exp\left( -\sigma \mathrm{Area} \right)$$

The rate of the exponential decay as the temporal extent increases goes with the *area* encircled by the Wilson loop. This behaviour is called *area-law* and is a criterium for confinement.

The Wilson loop is related to the *static potential* : the potential of two point sources infinitely heavy and separated by a distance $R$.

The static limit corresponds to an action where the spatial derivatives (spatial momenta) are neglected:

$$S_{stat}[\phi] = a^4 \sum_x \frac{1}{2} \left[ \left( \hat{\partial}_0^* \phi \right) \hat{\partial}_0 \phi + m^2 |\phi|^2 \right], \quad |\hat{\partial}_k \phi| \ll m\phi$$

the field values at different space points $\mathbf{x}$ are independent variables.

$$\langle \phi(\mathbf{x}, x_0) \phi^\dagger(\mathbf{y}, y_0) \rangle_\phi = \frac{a}{2 \sinh(a\omega)} e^{-(x_0 - y_0)\omega} \delta(\mathbf{x} - \mathbf{y}) U(\mathbf{x}, x_0; \mathbf{y}, y_0),$$

where $U(\mathbf{x}, x_0; \mathbf{y}, y_0)$ is the parallel transporter and $\cosh(a\omega) = 1 + \frac{1}{2} a^2 m^2$

The simplest gauge-invariant operator representing a $q\bar{q}$ pair separated by some spatial distance $|\mathbf{y} - \mathbf{x}| = R$ at time $t$ is

$$\mathcal{O}(t) = \phi^\dagger(\mathbf{y}, t) U(\mathbf{y}, t; \mathbf{x}, t) \phi(\mathbf{x}, t)$$

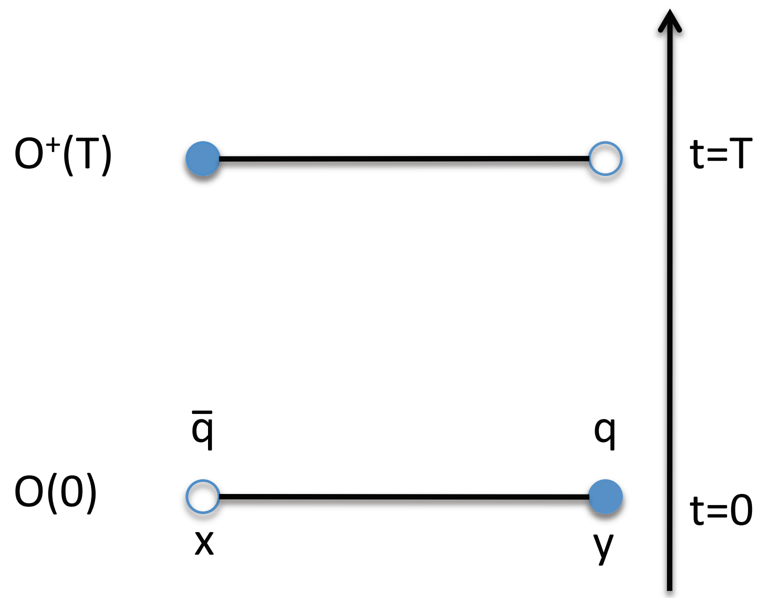
The correlator at large times $T \to \infty$,

$$C_{q\bar{q}}(T) \equiv \langle \mathcal{O}^\dagger(T) \mathcal{O}(0) \rangle_{\phi, U}$$

represents a $q\bar{q}$ pair separated by a distance $R$ that are created at time $x_0 = 0$ and evolve until time $T$.

The exponential decay in time of $C_{q\bar{q}}$ gives us information about the energy of this system

$$C_{q\bar{q}}(T) \sim \exp(-E(R)T), \quad E(R) = E_0 + V(R)$$




Integrating over the scalar fields first

$$C_{q\bar{q}}(T) = \left\langle \mathrm{Tr}\left[ U^\dagger(\mathbf{y},T;\mathbf{x},T)\langle\phi(\mathbf{y},T)\phi^\dagger(\mathbf{y},0)\rangle_\phi U(\mathbf{y},0;\mathbf{x},0)\langle\phi(\mathbf{x},0)\phi^\dagger(\mathbf{x},T)\rangle_\phi \right] \right\rangle_U \sim \langle W_{RT}\rangle,$$

neglecting $R$ independent factors.

$$\lim_{\beta \to 0} V(R) = \frac{R}{a^2} \log \left( \frac{2N^2}{\beta} \right) + ... = \sigma R + ...$$

This linear behaviour is a criterium for confinement, because the potential energy grows without bound when the quark and the antiquark are pulled apart.

$\sigma$ is called *the string tension*:

$$\lim_{\beta \to 0} \sigma = \frac{1}{a^2} \log \left( \frac{2N^2}{\beta} \right)$$

But no continuum limit: $\lim_{a \to 0} a^2 \sigma = \text{finite}$

The strong coupling analysis gets all the qualitative behaviour right, but there is no continuum limit in this approximation.

The existence of a continuum limit can be shown in the opposite limit of small coupling.

## Weak coupling expansion

Yang-Mills theories are perturbatively renormalizable. According to Wilson's renormalization group, this must imply that a continuum limit can be defined in lattice perturbation theory

On the lattice, the weak coupling expansion is a saddle-point expansion around the configurations with vanishing action:

$$U_\mu(x) = 1, \quad U_\mu(x) = \exp\left(-ig_0 a T^a A^a_\mu(x)\right)$$

It is necessary to fix the gauge if we are going to integrate over unbounded $A^a_\mu$.

The Fadeev-Popov procedure can be carried out almost identically on the lattice

$$S_{GF}[c,\bar{c},U] = S[U] + S_{meas}[U] + S_{FP}[c,\bar{c},U] + \frac{1}{2\alpha}\sum_a G^a(U)G^a(U)$$

## Feynman rules

A commonly used gauge is *Lorentz gauge*: $G[U] = \sum_\mu \hat{\partial}^*_\mu A_\mu(x)$

In momentum space: $A_\mu(p) = a^4 \sum_x e^{ip(x+a\frac{\hat{\mu}}{2})} A_\mu(x)$.

$$\hat{k}_\mu = \frac{2}{a} \sin\left(\frac{k_\mu a}{2}\right) \qquad \hat{k}^2 = \sum_\mu \hat{k}_\mu^2$$

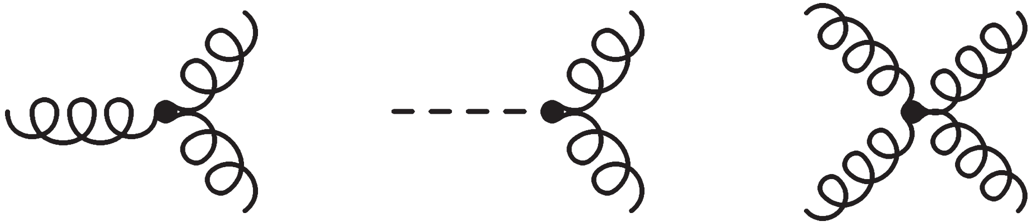
Gauge and ghost propagators:

$$-\frac{\delta_{ab}}{\hat{k}^2}\left[\delta_{\mu\nu} - (1-\alpha)\frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^2}\right]$$



$$\frac{\delta_{ab}}{\hat{k}^2}$$

At higher order in $g_0$ there are diagrams that have continuum analogs:



But also a gluon mass term or a two-gluon-two-ghost vertex:

1 loop divergences can be absorbed in $Z^{1/2} A_{\mu R} = A_\mu, \quad Z_g g_R = g_0$ Many miraculous cancellations take place to cancel disastrous contributions such as a gluon mass term or Lorentz-non-invariant terms. It can be shown to occur to all orders by an exact BRST invariance.

Reisz

Callan-Symanzik equations in the momentum subtraction scheme:

$$\Gamma^{(2)}(k)|_{k^2=\mu^2} = \text{tree} - \text{level}$$

$$\Gamma^{(4)}(k_1, k_2, k_3)|_{k_i k_j = \frac{1}{2}(3\delta_{ij}-1)\mu^2} = \text{tree} - \text{level},$$

where $\mu a \ll 1$.

$$g_R^2(\mu) = g_0^2 \left(1 - \frac{g_0^2}{16\pi^2}\frac{11 N_c}{3}(\log(a^2\mu^2) + c')\right)$$

$$\beta(g_0) \equiv -a \left.\frac{\partial g_0}{\partial a}\right|_{g_R \text{ fixed}} = -\beta_0 g_0^3 - \beta_1 g_0^5 + ... \quad \beta_0 = \frac{N_c}{16\pi^2}\frac{11}{3} > 0$$

$g_0 = 0$ is a zero of the $\beta$ function, i.e. a *UV fixed point*, therefore our target continuum limit corresponds to $g_0 = 0$

We can integrate the RG equation to get

$$a = c \exp\left(\frac{-1}{2\beta_0 g_0^2}\right)(g_0^2)^{-\frac{\beta_1}{2\beta_0^2}},$$

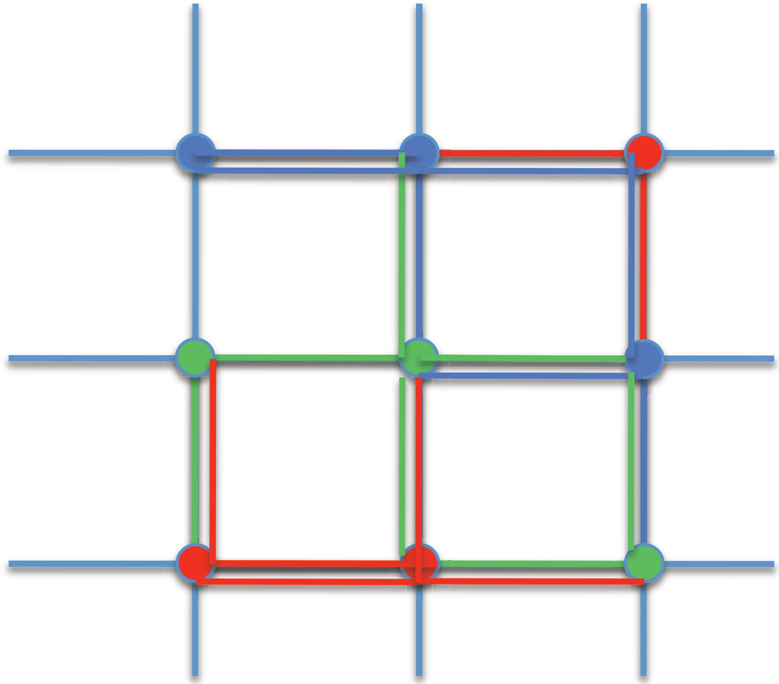
where $c$ is a constant of integration and does not depend on $a$

$\Lambda$ parameter

$$a\Lambda \equiv \exp\left(\frac{-1}{2\beta_0 g_0^2}\right)(\beta_0 g_0^2)^{\frac{-\beta_1}{2\beta_0^2}}$$

which remains constant in the continuum limit, and therefore all scales should be proportional to $\Lambda$ as we approach the continuum limit.

# Lecture IV: Lattice QCD

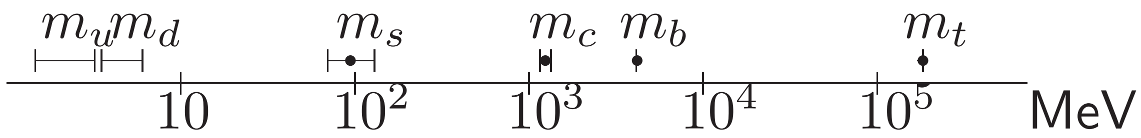


The original investigation on lattice field theory was motivated by the need to make predictions in QCD.

QCD is an $SU(3)$ gauge theory, with six flavours of quarks in the fundamental representation:

$$S_{QCD} = \int d^4x \sum_q \bar{\psi}_q(\gamma_\mu D_\mu + m_q)\psi_q - \frac{1}{2g_0^2}\mathrm{Tr}[G_{\mu\nu}G_{\mu\nu}]$$

Free parameters: the gauge coupling and six quark masses.



- Symmetries. At the classical level the symmetries of this action are

  - Lorentz invariance
  - $SU(3)$ gauge invariance
  - Discrete symmetries: C, P and T
  - Quark number: $\psi_q \to e^{i\alpha_q}\psi_q$

  In the absence of quark masses, there is a much larger global symmetry group: $U(6)_L \times U(6)_R$:

$$P_R\psi \to U_R P_R \psi \quad P_L\psi \to U_L P_L \psi \quad U_R, U_L \in U(6),$$

  But quark masses:

$$m_u \sim m_d \ll m_s \ll 1\ GeV < m_c \ll m_b \ll m_t$$

  The approximate flavour symmetry is at most $U(3)$ and not $U(6)$

- Spontaneous chiral symmetry breaking

  The chiral flavour group is broken to $U(3)_V$ spontaneously by a quark condensate

$$-\langle \bar{\psi}_i \psi_j \rangle \neq \Sigma \delta_{ij},$$

  invariant under $U_R = U_L = U_V \Rightarrow$ Nambu-Goldstone massless bosons, as many as generators have been broken with the quantum numbers of the pseudoscalar mesons: $\pi^{\pm}, \pi_0, K^{\pm}, K_0, \bar{K}_0, \eta$. One missing...

- Anomalous breaking of $U_A(1)$

$U(1)_A$ broken via an anomaly. At one loop the Noether current, $J_{5\mu} = \sum_q \bar{\psi}_q \gamma_\mu \gamma_5 \psi_q$, is not conserved

$$\partial_\mu J_{5\mu} = \frac{g_0^2}{16\pi^2} \epsilon_{\alpha\beta\gamma\delta} \mathrm{Tr}[F_{\alpha\beta} F_{\gamma\delta}],$$

Lattice QCD can (eventually) solve QCD from *first principles*

## Wilson formulation of Lattice QCD

$$S_{QCD}[U,\bar{\psi},\psi] = S[U] + S_W[U,\bar{\psi},\psi]$$

$$\begin{aligned} S[U] &\equiv \frac{2}{g_0^2}\sum_x \sum_{\mu<\nu} \mathrm{Tr}\left[1 - \frac{1}{2}\left(U_{\mu\nu}(x) + U^\dagger_{\mu\nu}(x)\right)\right] \\ S_W[U,\bar{\psi},\psi] &\equiv a^4 \sum_{q,x} \bar{\psi}_q \left[D_W + m_q\right]\psi_q, \end{aligned}$$

It is common practice to rewrite the fermionic action in terms of the parameter $\kappa$:

$$
\begin{aligned}
S_W &= a^4 \sum_{q,x} \bar{\psi}_q(x)\psi_q(x) - \kappa_q \left( \sum_{q,x,\mu} \bar{\psi}_q(x)(\gamma_\mu - r)U_\mu(x)\psi_q(x + a\hat{\mu}) \right. \\
&+ \left. \bar{\psi}_q(x)(\gamma_\mu + r)U^\dagger_\mu(x - a\hat{\mu})\psi_q(x - a\hat{\mu}) \right),
\end{aligned}
$$

$$
\kappa_q \equiv \frac{1}{2am_q + 8r}.
$$

In the free case, the massless limit corresponds to the critical value $\kappa_c = \frac{1}{8r}$.

The partition function is

$$\mathcal{Z} = \int dU\, d\bar{\psi} d\psi e^{-S_{QCD}[U,\bar{\psi},\psi]} \equiv \int dU\, \mathcal{Z}_F[U] e^{-S_g[U]}$$

$$\mathcal{Z}_F[U] \equiv \int d\bar{\psi} d\psi e^{-S_W[U,\bar{\psi},\psi]} = \prod_q \det\left(D_W + m_q\right)$$

- Positivity of the transfer matrix and Hilbert space interpretation: consequence of the property for the pure gauge and the free fermion cases ($r \leq 1$)

- Renormalizability can be shown to hold to all orders.

For any correlation function involving fermion fields, the integration over Grassmann variables can always be done analytically:

$$\langle \psi_{\alpha,i}(x)\bar{\psi}_{\beta,j}(y)\rangle = \mathcal{Z}^{-1}\int DU \langle \psi_{\alpha,i}(x)\bar{\psi}_{\beta,j}(y)\rangle_F \prod_q \det\left(D_W + m_q\right)\ e^{-S_g[U]},$$

$$\langle \psi(x)_{\alpha i}\bar{\psi}(y)_{\beta j}\rangle_F = \delta_{ij}\left[(D_W + m_i)^{-1}\right]^{\alpha\beta}_{xy}.$$

All fermion integrals result in product of propagators.

Integration over $U$ by importance sampling methods

$$\langle \mathcal{O}[U]\rangle = \int DU \mathcal{O}[U]\ P[U], \quad P[U] \geq 0, \quad \int D[U]P[U] = 1$$

$$\mathcal{O}[U] \rightarrow \text{e.g. Product of propagators}, \quad P[U] \sim \prod_q \det\left(D_W + m_q\right),\ e^{-S_g[U]}$$

Representative emsemble: random sequence $\{U_1, ..., U_N\}$ distributed according to $DU P[U]$:

$$\langle \mathcal{O}[U] \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i] + \mathcal{O}\left( \frac{1}{\sqrt{N}} \right)$$

- Numerical evaluation of the propagator iteratively

$$(D_W + m)\psi(x) = \eta(y) \rightarrow \psi(x) = (D_W + m)^{-1} \eta(y)$$

harder the larger the condition number $(\lambda_{max}/\lambda_{min})$

- Hybrid montecarlo to general the representative emsemble

Duane et al.

Much of the progress in the field of recent years is due to improvement in the algorithms

Lüscher, Les Houches lectures

# Continuum Limit

The question is then: what are the relevant or marginal operators that need to be tuned in the continuum limit ?

As a result of the breaking of chiral symmetry the term $\bar{\psi}\psi$ becomes relevant. If chiral symmetry would be broken softly, it would be $m\bar{\psi}\psi$ and marginal

$$g_0 \to 0, \qquad \kappa^q \to \kappa_c^q$$

A very useful procedure to define the massless point, beyond perturbation theory, is to impose the PCAC relation.

In the functional formulation, symmetries $\leftrightarrow$ relations between correlation functions: *Ward-Takahashi identities*

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon_a(x)\delta_a\phi(x),$$

$$\frac{\partial}{\partial x_\mu}\langle\phi(x_1)\phi(x_2)...\phi(x_n)\mathcal{J}^a_\mu(x)\rangle = \langle\phi(x_1)\phi(x_2)...\phi(x_n)\left.\frac{\delta\mathcal{L}}{\delta\epsilon_a(x)}\right|_{\epsilon=0}\rangle$$
$$-\sum_i \delta(x_i - x)\langle\phi(x_1)..\delta_a\phi(x_i)..\phi(x_n)\rangle,$$

where

$$\mathcal{J}^a_\mu(x) \equiv \left.\frac{\delta\mathcal{L}(\phi+\epsilon^a\delta_a\phi)}{\delta\partial_\mu\epsilon_a(x)}\right|_{\epsilon=0}.$$

is the conserved Noether current.

Impose chiral WI to hold in the continuum limit

Bochicchio et al

Consider the following non-singlet transformation $\mathrm{Tr}[T^a] = 0$

$$\delta\psi(x) \to i\epsilon_a(x) T^a \gamma_5 \psi(x),$$
$$\delta\bar{\psi}(x) \to i\epsilon_a(x) \bar{\psi}(x) T^a \gamma_5.$$

Performing such a change of variables in the functional integral we get the lattice WI:

$$\langle O(y)\ \hat{\partial}^*_\mu A^a_\mu(x)\rangle = \langle O(y)\ \bar{\psi}(x)\gamma_5\{M, T^a\}\psi(x)\rangle + \langle O(y) X^a(x)\rangle - i\left\langle \frac{\delta O(y)}{\delta\epsilon^a(x)} \right\rangle.$$

$$X^a(x) = \delta_\epsilon(\mathrm{Wilson\ term}).$$

Naive continuum limit: $X^a \to 0$, while $A_\mu(x) \to$ to the continuum axial current.

The anomalous term, $X^a$ induces UV divergences at higher orders:

$$X^a = -2m_c P^a - (Z_A - 1)\hat{\partial}^*_\mu A_\mu + X^a_R,$$

where the last term is a renormalized operator that vanishes in the continuum limit and $m_c$ and $Z_A - 1$ are the mixing coefficients of $X^a$ with the lower dimensional operators.

$$\lim_{a \to 0} \langle O(y) Z_A \partial^*_\mu A^a_\mu \rangle = \lim_{a \to 0} \langle O(y) \bar{\psi}(x) \gamma_5 \{M - m_c, T^a\} \psi(x) \rangle - i \left\langle \frac{\delta O(y)}{\delta \epsilon^a(x)} \right\rangle .$$

In the continuum limit we recover the standard chiral WI, provided we find the values $Z_A$ and $m_c$.

In summary, the consequence of the explicit chiral symmetry breaking by the Wilson term is twofold:

- The mass needs to be tuned towards $m_c$, where $m_c$ can be found, for example, from a linear fit of the ratio:

$$\frac{\langle \hat{\partial}^*_\mu A^a_\mu(x) P^a(0)\rangle}{\langle P^a(x) P^a(0)\rangle} \sim M - m_c \equiv m_{PCAC}$$

- The axial current is renormalized.

- When considering operators such as the 4-fermion weak operators the mixing pattern is much more complicated

## Observables

- Spectrum from the KL representation of two-point correlation functions of appropriately chosen operators at large Euclidean times

- Decay constants from pion to vacuum matrix elements of the axial current

- Form factors:one particle state matrix elements of density operators: vector, axial, scalar, ...

- Two-body decays

- ...

## Low-lying Spectrum

How do we choose the operator ?

Operators with a Hilbert interpretations are products of the fundamental fields $\psi, \bar{\psi}$ and the spatial plaquettes at fixed times: any operator in the Hilbert space can be represented by creation and annihilation operators that create the one-particle asymptotic states in the interacting theory.

If quantum numbers are the right ones (spin, color, isospin, parity, etc) the operator will generically have an overlap with the one-particle state we are interested in

We do not know a priori which operator maximizes this overlap and there are several techniques to improve it: *variational techniques, smearing,* etc

## Mesons

The simplest operators that are used to compute meson correlation functions are of the form:

$$M^a(x) \equiv \bar{\psi}_{\alpha i c}(x)\Gamma_{\alpha\beta}T^a_{ij}\psi_{\beta j c}(x)$$

$\Gamma = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, ...\}$ for the scalar, axial, vector and axial vector...

$T^a$ is a matrix in flavour space that fixes the flavour quantum numbers

Color singlet

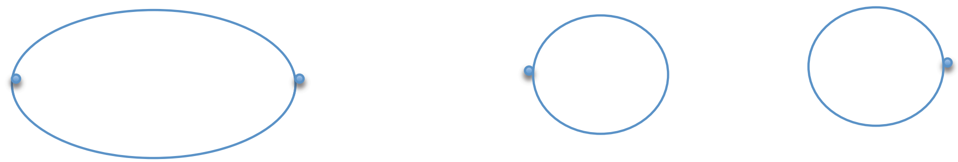
In order to improve the signal it is common practice to project on the zero spatial momentum states by computing the correlator

$$C_M(x_0) = \sum_{\mathbf{x}} \langle M^a(x_0, \mathbf{x}) M^a(0, \mathbf{0}) \rangle$$

Grassmann integrations can be readily performed

$$
\begin{aligned}
C_M(x_0) &= \frac{1}{\mathcal{Z}} \int DU e^{-S_g[U]} \det(D_W + M) \sum_{\mathbf{x}} \\
& \quad \{ -\mathrm{Tr}[(D_W + M)^{-1}_{0,x} (\Gamma \otimes T^a) (D_W + M)^{-1}_{x,0} (\Gamma \otimes T^b)] \\
&+ \mathrm{Tr}[(D_W + M)^{-1}_{0,0} (\Gamma \otimes T^a)] \mathrm{Tr}[(D_W + M)^{-1}_{x,x} (\Gamma \otimes T^b)] \}
\end{aligned}
$$

The two terms correspond to the connected and disconnected contributions:



Disconnected contributions much harder to compute numerically because the sum over $\mathbf{x}$ requires the inversion of the Dirac operator $(L/a)^3$ times, while the connected contribution can be obtained with a single inversion per spin and colour.

## Baryons

Baryons are $qqq$ color singlets. We can take the following operators:

$$B^{abc}_{\alpha\beta\gamma} = \psi(x)_\alpha \equiv \epsilon_{c_1c_2c_3}\psi_{\alpha a c_1}\psi_{\beta b c_2}\psi_{\gamma c c_3},$$

where $a, b, c$ are the flavour indices and $\alpha, \beta, \gamma$ the spinor ones. The contraction of these three quark object with appropriate tensors ensure the right flavour content and spin

*Example:* the proton is a $J = 1/2$, $P = +1$ and $I = 1/2$ $uud$ state

First combine the $d$ and one $u$ in a $J = 0$, $I = 0$ diquark state and then add the third one. We need therefore to combine the $u$ and $d$ antisymmetrically both in flavour and spin :

$$(u_\alpha d_\beta - d_\alpha u_\beta)(C\gamma_5)_{\alpha\beta},$$

where $C\gamma_5$ is antisymmetric.

The two terms are the same and the possible proton operator is given by

$$p_\gamma = u_\gamma u_\alpha d_\beta (C\gamma_5)_{\alpha\beta} = u^T C\gamma_5 d u_\gamma,$$

where the color indices are contracted with the $\epsilon$ tensor.

The corresponding anti-proton is

$$\bar{p}_\gamma = \bar{d} C\gamma_5 \bar{u}^T \bar{u}_\gamma$$

The two-point correlation functions of those operators at large $x_0$ separation, are dominated by the lightest one-particle states in the corresponding channel:

$$\lim_{x_0\to 0} \sum_{\mathbf{x}} \langle B(x)B(0)\rangle = \lim_{x_0\to 0} \sum_{\mathbf{x}} \langle 0|T(\hat{B}(x)\hat{B}(0))|0\rangle_E = \frac{Z_L}{2}(1+\gamma_0)\, e^{-m_L x_0}$$

where $m_L$ is the mass of the lightest state in this channel, $|L\rangle$, and $Z_L = |\langle 0|\hat{B}(0)|L\rangle|^2$, the vacuum-to-this-state matrix element.

## Decay constants: pion to vacuum matrix elements

Chiral Ward indentity $\Rightarrow$ the axial current couples to the one pseudoscalar meson states, and the lightest of them is the pion $|\pi\rangle$:

$$\langle 0|A_\mu^a(x)|\pi(p)\rangle = iF_\pi p_\mu e^{-ipx}$$

$F_\pi$ can be determined from the normalization of the axial-current two-point correlator provided it is appropriately renormalized:

$$-\lim_{x_0\to\infty} Z_A^2 \sum_{\mathbf{x}} \langle A_0(x)A_0(0)\rangle = \frac{F_\pi^2 M_\pi}{2}\exp(-M_\pi x_0)$$

A very essential requirement is therefore to have $Z_A$.

Phenomenology: the leptonic decays widths of pseudoscalar mesons $M \to \bar{\nu}_l l$, from which several of the elements of the CKM matrix are best determined.

## Form factors: single state matrix elements of current operators

Meson semileptonic decays in which a meson decays into a lighter one emiting two leptons, e.g. $B \to \pi l \nu_l$ (important in the determination of $V_{ub}$) depends on the matrix element of the weak current between the initial and final meson states:

$$\langle M|\bar{q}T^a\gamma_\mu(1-\gamma_5)q|M'\rangle,$$

where the flavour quantum numbers of $M, M'$ and $T^a$ should be appropriately fixed for the given process.

LSZ reduction formulae: related to the expectation value of the time ordered product of three operators: the vector current and the two operators that have an overlap with the initial and final meson states,

$$\lim_{x_0,y_0\to+\infty,-\infty}\sum_{\mathbf{x},\mathbf{y}}\langle M^a(x)J^b_\mu(0)M^c(y)\rangle$$

In contrast with two point functions that depend on a single momentum, the three-point functions depend on two and therefore the matrix element has a non-trivial momentum dependence dictated by Lorentz invariance:

$$\langle\pi(p)|J_\mu(q)|B(p')\rangle = f^+(q^2)\left[p' + p - \frac{m_B^2 - m_\pi^2}{q^2}q\right]_\mu + f^0(q^2)\frac{m_B^2 - m_\pi^2}{q^2}q_\mu$$

$f^+(q^2)$, $f^0(q^2)$ are called *form factors* and in principle they must be determined in the whole kinematical range of $q^2$.

## Two-body decays

Other processes such as $K \to \pi\pi, \rho \to \pi\pi$, etc involve also three-point functions, however their large time behaviour does not contain sufficient information to reconstruct the corresponding $S$-matrix element

Maiani-Testa theorem

It is important to point out that there is nothing wrong with LSZ reduction formula on the Euclidean infinite lattice

Lüscher

Any $S$-matrix element can be computed by:

- computing the connected Euclidean correlation functions in momentum space

$$\sum_{x_n} ... \sum_{x_1} e^{-iq_1 x_1} ... e^{-iq_n x_n} \langle O(x_1) .... O(x_n) \rangle = S_n(q_1, ..., q_n),$$

- Wick rotating them back to Minkowski:

$$W_n(E_1, ...E_n) = S_n(q_1, ..., q_n)|_{q_i^0=(i-\epsilon)E_i}$$

- The S-matrix element is the given by

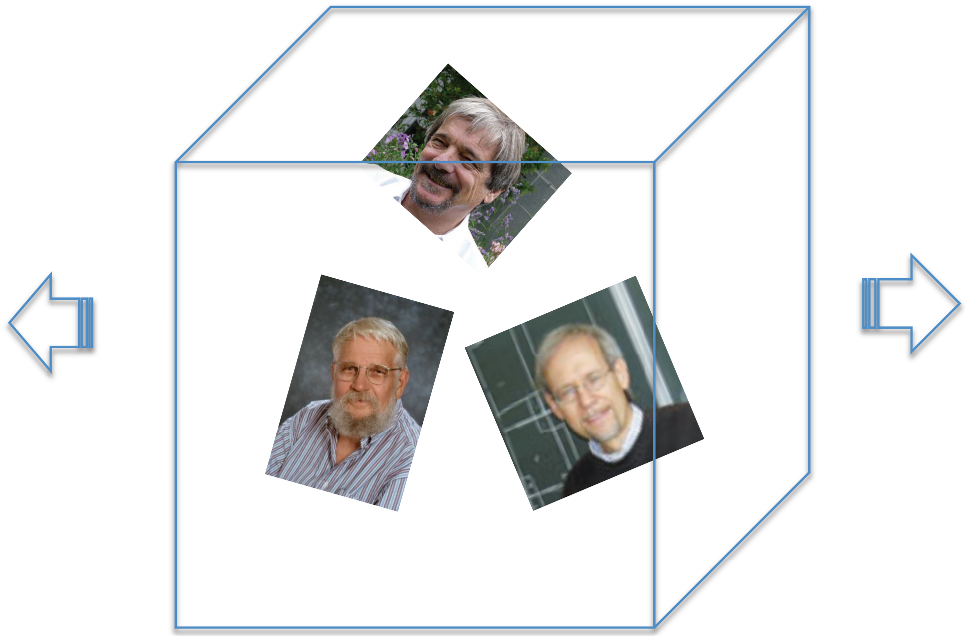
$$\langle \mathbf{p}_3, ..., \mathbf{p}_n; out|\mathbf{p_1}\mathbf{p}_2\rangle = \prod_k \frac{(E_k^2 - \omega(\mathbf{p}_k))}{\sqrt{Z_k}} W_n \Bigg|_{E_i=\pm\omega(\mathbf{p}_i)}$$

This method is however numerically hopeless. There are smarter ways to go around, by using finite-size scaling techniques.

QCD in a box is a wonderful laboratory from which physical information can be extracted

QCD in a box is a wonderful laboratory from which physical information can be extracted

*Bernese in a box...*



- Finite-size dependence of one particle masses is related to the forward elastic scattering amplitude

Lüscher

- Two particle spectra in a box is related to the scattering phase shifts and unstable particle widths

Lüscher

- The Nambu-Goldstone bosons in a box behave in a way that can be predicted by Chiral Perturbation Theory and provide a different regime to match QCD with the chiral Lagrangian: the so-called $\epsilon$-regime

Gasser,Leutwyler

- Non-perturbative renormalization: the renormalization scale is set by the box size. Implemented in the Schrödinger functional scheme

Alpha col.

and the list is probably not exhausted...

An important message is that in lattice QCD simulations the optimal conditions to extract physical parameters are not necessarily the same conditions as in real experiments.

The universality of our results needs $a \to 0$, but we should also exploit as much as possible the possibilities that the lattice offers of probing QCD in new conditions (unphysical quark masses, finite volume, etc...)

# Thank you!